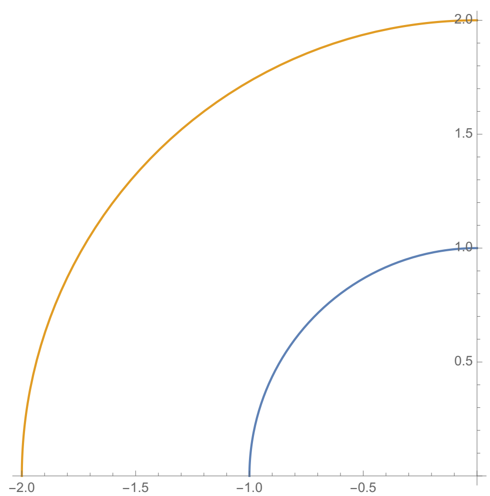
Math 210 (Lesieutre)
13.3: Double integrals in polar coordinates
March 3, 2017

Problem 1. *Sketch the regions corresponding to the following double integrals.*

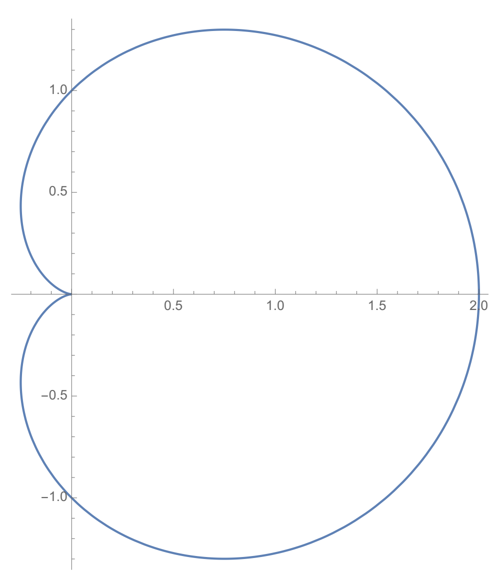
a) $\int_1^2 \int_{\pi/2}^{\pi} f(r,\theta)\, r\, dr\, d\theta$

It's the region between orange and blue curves in the following illustration, which is a sort of quarter-ring.



b) $\int_0^{1+\cos\theta} \int_0^{2\pi} f(r,\theta)\, r\, dr\, d\theta$

This is the inside of the cardioid pictured inthe following picture.



Problem 2. *Find the area inside the cardioid* $r(\theta) = 1 + \cos\theta$.

This is a classic one. To find the area, we have to integrate $1\,dA = 1\,r\,drd\theta$ on the region in question. Don't forget the "r"!. This is

$$\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta.$$

The inner integral is:

$$\int_0^{1+\cos\theta} r\,dr = \frac{r^2}{2}\Big|_0^{1+\cos\theta} = \frac{(1+\cos\theta)^2}{2} = \frac{1+2\cos\theta+\cos^2\theta}{2}$$
$$= \frac{1}{2} + \cos\theta + \frac{1}{2}\left(\frac{1+\cos 2\theta}{2}\right) = \frac{3}{4} + \cos\theta + \frac{1}{4}\cos 2\theta.$$

The full integral is now given by

$$\int_0^{2\pi} \frac{3}{4} + \cos\theta + \frac{1}{4}\cos 2\theta\,d\theta = \frac{3}{4}(2\pi) + 0 + 0 = \frac{3\pi}{2}.$$

Seems plausible.

Problem 3. *Find the volume of the region under the graph $f(x,y) = 1 - x^2 - y^2$ above the unit circle R.*

The volume is going to be given by $\iint_R f(r,\theta)\,dA$. There are three things we need to do: (i) express the bounds on the integral in polar (ii) express the function in question in polar (iii) express the area element dA in polar. Then we need (iv): actually compute the integral.

The function $f(x,y) = 1-x^2-y^2$ is given in terms of θ by $f(r,\theta) = 1-r^2$. The area element is $dA = r\,dr\,d\theta$ (that's what you always want to use here). The volume is then given by

$$\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta.$$

The inner integral is

$$\int_0^1 (1-r^2)\,r\,dr = \int_0^1 (r-r^3)\,dr = \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{1}{4}.$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{4}\,d\theta = (2\pi)\frac{1}{4} = \frac{\pi}{2}.$$

That's the volume. Seems plausible.

Problem 4. a) *Rewrite the following integral in Cartesian coordinates:* $\iint_R 2xy\,dA$*, where* R *is part of a circle of radius* 4 *that lies in the first quadrant.*

Again, we need to get the bounds in polar, the function in polar, and dA is polar. The hardest part is the function. Remember that $x = r\cos\theta$ and $y = r\sin\theta$, and so $f(x, y) = 2xy = 4r^2\cos\theta\sin\theta = 2r^2\sin 2\theta$. This gives us

$$\int_0^{\pi/2}\int_0^4 2r^2\sin(2\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^4 2r^3\sin(2\theta)\,dr\,d\theta$$

b) *Evaluate the integral.*

This one is

$$\begin{aligned}\int_0^{\pi/2}\int_0^4 2r^3\sin(2\theta)\,dr\,d\theta &= \left(\int_0^{\pi/2}\sin 2\theta\,d\theta\right)\left(\int_0^4 2r^3\,dr\right)\\ &= \left(\frac{-\cos 2\theta}{2}\bigg|_0^{\pi/2}\right)\left(\frac{r^4}{2}\bigg|_0^4\right) = \left(\frac{1}{2} - \frac{-1}{2}\right)\left(\frac{256}{2} - 0\right) = 128.\end{aligned}$$