Math 210 (Lesieutre)
13.5: Triple integrals in cylindrical coordinates
March 14, 2017

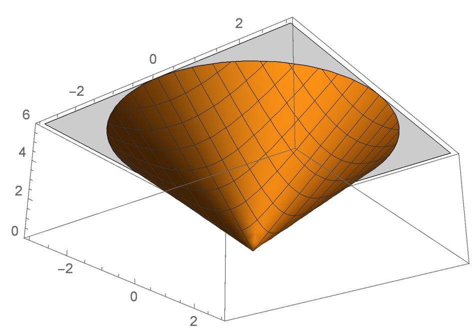
Problem 1. a) *Find the cylindrical coordinates for the point* $(x, y, z) = (0, 1, 1)$.
We have $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = 1$, $z = 1$, and $\theta = \pi/2$ (same as polar).

b) *Find the rectangular coordinates for the point* $(1, \pi/4, 3)$.
We get

$$\begin{aligned} x &= r\cos\theta = (1)(\sqrt{2}/2) = \sqrt{2}/2, \\ y &= r\sin\theta = (1)(\sqrt{2}/2) = \sqrt{2}/2, \\ z &= 3, \end{aligned}$$

and so the point is $(\sqrt{2}/2, \sqrt{2}/2, 3)$.

c) *Sketch the cylindrical surface defined by* $z = 2r$.
This is a cone with vertex at the origin:



Problem 2. *Use a triple integral in cylindrical coordinates to compute the volume of a cylinder with radius* R *and height* h.

To find the area of a region, you integrate the function 1. Ditto to find the volume of a region in 3d: we need to integrate the function 1 over this region. We are going to need $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq h$. The integral then becomes

$$\begin{aligned} V &= \int_0^R \int_0^{2\pi} \int_0^h 1 \, dV \\ &= \int_0^R \int_0^{2\pi} \int_0^h 1 \, r \, dz \, d\theta \, dr \end{aligned}$$

Inner:

$$\int_0^h r \, dz = rh$$

Middle:

$$\int_0^{2\pi} rh\,d\theta = 2\pi rh$$

Outer:

$$\int_0^R 2\pi rh\,dr = \pi r^2 h\Big|_0^R = \pi R^2 h.$$

That's the right answer.

Problem 3. *You want to integrate the function* $x^2+y^2+z^2$ *over a cone with base a circle of radius* 3 *in the* xy*-plane centered at the origin, and vertex at the rectangular point* $(0,0,6)$*. Set up the corresponding integral in cylindrical coordinates.*

This is a little more interesting, because one of the bounds depends on the other: the range of z's that we integrate over depends on the value of r. We need $0 \le \theta \le 2\pi$ and $0 \le r \le 3$. For a given r, the maximum z is $6-2r$: this gives 6 at $r=0$ and 0 at $r=3$.

- What are the bounds? It's $0 \le \theta \le 2\pi$, $0 \le r \le 3$, and $0 \le z \le 6-2r$.
- What's the function? $x^2+y^2+z^2 = r^2+z^2$.
- What's dv? In spherical, $dv = r\,dr\,d\theta\,dz$

Thus the integral is

$$\int_0^{2\pi}\int_0^3\int_0^{6-2r} (r^2+z^2)\,r\,dz\,dr\,d\theta.$$

Problem 4. *Find the volume of the solid bounded by the paraboloid* $z = 4-x^2-y^2$ *and the* xy*-plane.*

The paraboloid intersects the xy-plane where $4-x^2-y^2=2$, which is a circle of radius 2 We have $0 \le \theta \le 2\pi$, $0 \le r \le 2$, and $0 \le z \le 4-r^2$. The volume then works out to be

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} 1\,r\,dz\,dr\,d\theta.$$

Inner:

$$\int_0^{4-r^2} r\,dz = r(4-r^2) = 4r-r^3,$$

(since we're integrating a constant dz) Middle:

$$\int_0^2 4r-r^3\,dr = \left(2r^2-\frac{r^4}{4}\right)\Bigg|_0^2 = 4-0=4.$$

Outer:

$$\int_0^{2\pi} 4\,d\theta = 8\pi.$$

Problem 5. *Consider the integral*

$$\int_0^2 \int_0^{\pi/4} \int_0^3 r^3 z \, dz \, d\theta \, dr.$$

a) *Sketch the region of integration.*

Take a cylinder with radius 2 and height 3, and then take the part that's above the 1st quadrant in the xy-plane. That's your region.

b) *Convert this to an integral in rectangular coordinates.*

The bounds are going to be

$$\int_{z=0}^{3} \int_{y=0}^{\sqrt{4-x^2}} \int_{x=0}^{2} dx \, dy \, dz$$

We also need to convert the volume element: $r \, dz \, d\theta \, dr = dx \, dy \, dz$. So when we switch over, we are going to lose a factor of r from the integrand. The function is then $r^2 z$, which is $(x^2 + y^2)z$. The integral is therefore

$$\int_{z=0}^{3} \int_{y=0}^{\sqrt{4-x^2}} \int_{x=0}^{2} (x^2 + y^2) z \, dx \, dy \, dz$$