Math 210 (Lesieutre)
Exam 2 review
March 15, 2017

Problem 1. *Find the point on the plane* $4x + 3y + z = 10$ *nearest to* $(2, 0, 1)$.

The first thing to realize is that this is really a constrained optimization problem. The distance from (x, y, z) to $(2, 0, 1)$ is just $\sqrt{(x-2)^2 + y^2 + (z-1)^2}$. We want to minimize this, subject to the constraint $4x + 3y + z = 10$. As usual, it will be simpler to minimize the square of the distance, which will find the correct closest point but save us a lot of trouble on the algebra. So we are going to use Lagrange multipliers with

$$f(x, y, z) = (x-2)^2 + y^2 + (z-1)^2$$
$$g(x, y, z) = 4x + 3y + z - 10.$$

The gradients are

$$\nabla f = \langle 2x - 4, 2y, 2z - 2 \rangle$$
$$\nabla g = \langle 4, 3, 1 \rangle .$$

We need $\nabla f = \lambda \nabla g$, so

$$\langle 2x - 4, 2y, 2z - 2 \rangle = \lambda \langle 4, 3, 1 \rangle .$$

We get four equations in four variables:

$$\begin{aligned} 2x - 4 &= 4\lambda \\ 2y &= 3\lambda \\ 2z - 2 &= \lambda \\ 4x + 3y + z &= 10. \end{aligned}$$

As usual this isn't very fun to solve, but we can do it. There are many approaches to the algebra. I'm going to replace all three variables in the last equation with λ:

$$\begin{aligned} 4x &= 8 + 8\lambda \\ 3y &= \frac{9}{2}\lambda \\ z &= 1 + \frac{1}{2}\lambda \\ 4x + 3y + z &= 9 + 13\lambda = 10, \end{aligned}$$

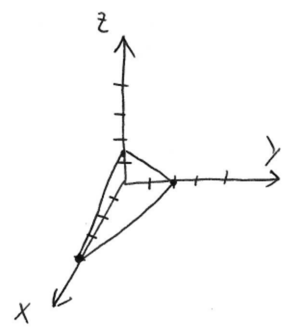
so $\lambda = \frac{1}{13}$. Plugging this back in to the original equations, we end up with

$$x = \frac{28}{13}, \quad y = \frac{3}{26}, \quad z = \frac{27}{26}.$$

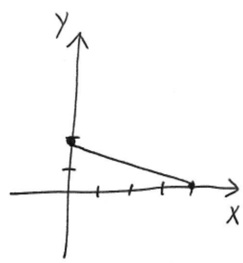
So that's our closest point.

Problem 2. *Write down an iterated triple integral that expresses the volume of the tetrahedron bounded by the xy-plane, the yz-plane, the xz-plane, and the plane $2x + 4y + 6z = 8$. Do not evaluate the integral.*

This one is lifted from an old exam. First step is to draw the thing. You can use the traces like we did in class, but with planes often the best strategy is to plot the points where it intersects the axes. It meets the x-axis where $y = 0$ and $z = 0$, so that $2x = 8$, i.e $x = 4$. This is the point $(4, 0, 0)$. Similarly it goes through $(0, 2, 0)$ and $(0, 0, 4/3)$. So it looks something like the plane sketched below:



To set up the integral, we should first figure out the bounds on x and y that give the base of the tetrahedron. The base is the triangle in the xy-plane sketched below:



What's the equation for the line? Well, this is the line with $z = 0$, so $2x + 4y = 8$, which amounts to $y = 2 - \frac{x}{2}$. From the picture, we can see that x goes from 0 to 4. For a given value of x, y goes from 0 to $2 - \frac{x}{2}$.

At last, we need the bounds on z. For given (x, y), the lower bound is 0 (since the tetrahedron has base on the xy-plane), and the upper bound is the plane in question. The formula for z we get for the plane is $z = \frac{1}{6}(8 - 2x - 4y) = \frac{1}{3}(4 - x - 2y)$.

We want to integral the function 1 to get volume, and $dV = dx\, dy\, dz$. So our volume is

$$V = \iiint_D dV = \int_{x=0}^{4} \int_{y=0}^{2-\frac{x}{2}} \int_{z=0}^{\frac{1}{3}(4-x-2y)} dz\, dy\, dx.$$

Problem 3. *Find the maximum value of the function x^2y such that (x, y) lies on the unit circle.*

We are trying to maximize $f(x, y) = x^2y$ subject to the constraint $x^2 + y^2 - 1 = 0$, so $g(x, y) = x^2 + y^2 - 1$. Then $\nabla f = \langle 2xy, x^2 \rangle$ and $\nabla g = \langle 2x, 2y \rangle$. The Lagrange multiplier equation is $\nabla f = \lambda \nabla g$, so $2xy = 2x\lambda$ and $x^2 = \lambda 2y$. In addition to these two equations, we have the third equation $x^2 + y^2 - 1 = 0$.

Now, if x is not 0, the first equation just says $y = \lambda$, and the second then gives $x^2 = 2y^2$. Plugging this into the third equation, $2y^2 + y^2 - 1 = 0$, so $y^2 = 1/3$, and we have $y = \pm 1/\sqrt{3}$. Then $x^2 = 2/3$, so $x = \pm 2/\sqrt{3}$.

It could be that $x = 0$ instead. Then the first equation will hold no matter what, and the third says that $y = \pm 1$. The second equation is then satisfied if $\lambda = 0$. Altogether there are six points we need to worry about:

(x, y)	$f(x, y)$
$(2/\sqrt{3}, 1/\sqrt{3})$	$2/3\sqrt{3}$
$(2/\sqrt{3}, -1/\sqrt{3})$	$-2/3\sqrt{3}$
$(-2/\sqrt{3}, 1/\sqrt{3})$	$2/3\sqrt{3}$
$(-2/\sqrt{3}, -1/\sqrt{3})$	$-2/3\sqrt{3}$
$(0, 1)$	0
$(1, 0)$	$0.$

The maxima are $(\pm 2/\sqrt{3}, 1/\sqrt{3})$, where the value of the function is $\frac{2}{3\sqrt{3}}$.

Problem 4. a) *What is the tangent plane to the surface* $x^2+y^2+z^2 = 9$ *at the point* $(1, 2, 2)$*?*
The tangent plane is normal to the gradient of the function defining the surface: $f(x, y, z) = x^2 + y^2 + z^2 - 9$. The gradient is $\nabla f = \langle 2x, 2y, 2z \rangle$, which at the point in question is $\nabla f(1, 2, 2) = \langle 2, 4, 4 \rangle$. So the plane is normal to this vector. Since the tangent plane passes through the point $(1, 2, 2)$, it must be

$$2(x - 1) + 4(y - 2) + 4(z - 2) = 0.$$

b) *Consider the function* $f(x, y) = x\sqrt{y}$. *Use a linear approximation centered at* $(1, 1)$ *to approximate the value of* $f(1.1, 1.2)$.
We have

$$\begin{aligned} f_x &= \sqrt{y} \\ f_y &= \frac{x}{2\sqrt{y}}. \end{aligned}$$

At the point $(1, 1)$, these things evaluate to

$$\begin{aligned} f(1, 1) &= 1 \\ f_x(1, 1) &= 1 \\ f_y(1, 1) &= \frac{1}{2}. \end{aligned}$$

The formula for linear approximation says that

$$\begin{aligned} f(x,y) &\approx f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= 1 + 1(x-1) + \frac{1}{2}(y-1). \end{aligned}$$

Plugging in $(x,y) = (1.1, 1.2)$ as requested, we get

$$f(1.1, 1.2) \approx 1 + 1(1.1-1) + \frac{1}{2}(1.2-1) = 1 + 0.1 + 0.1 = 1.2.$$

The true value is 1.20499. Not bad.

Problem 5. *Let C be a cylinder between $z = 1$ and $z = 3$ with radius 2 and centered around the z-axis, and let R be the portion of this cylinder that lies above the second quadrant in the xy-plane. Suppose you want to integrate the function $2xyz$ over R. Set up the corresponding integral in cylindrical coordinates (you need not evaluate it).*

We have $1 \le z \le 3$, $0 \le r \le 2$, and $\pi/2 \le \theta \le \pi$. The volume element is $dV = r\,dr\,d\theta\,dz$. The function is

$$2xyz = 2(r\cos\theta)(r\sin\theta)z = 2r^2\cos\theta\sin\theta z = r^2\sin(2\theta)z.$$

So the integral is...

$$\int_{z=1}^{3}\int_{0}^{2}\int_{\pi/2}^{\pi} r^2 z\sin(2\theta)\, r\,d\theta\,dr\,dz$$

Problem 6. *Consider the double integral*

$$\int_{x=0}^{1}\int_{y=x^2}^{x} 2y\,dy\,dx.$$

a) *Evaluate the integral directly.*

Inner:

$$\int_{y=x^2}^{x} 2y\,dy = y^2\Big|_{y=x^2}^{x} = (x)^2 - (x^2)^2 = x^2 - x^4.$$

Outer:

$$\int_{x=0}^{1} x^2 - x^4\,dx = \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

b) *Sketch the region of integration, and switch the order of integration. (You do not need to evaluate the new integral.)*

Here's the region:

In the other order of integration, this works out to be

$$\int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} 2y \, dx \, dy.$$

(You don't have to evaluate it, but I did, and I got 2/15 this way as well, which is a good sign.)

Problem 7. *Consider the function* $f(x, y) = x^2 - 2x + y^2$.

a) *Find the critical points of* $f(x, y)$ *and classify the types of each.*
We have

$$\begin{aligned} f_x(x, y) &= 2x - 2 \\ f_y(x, y) &= 2y \\ f_{xx}(x, y) &= 2 \\ f_{xy}(x, y) &= 0 \\ f_{yy}(x, y) &= 2. \end{aligned}$$

The critical point is where $2x - 2 = 0$ and $4y^3 = 0$. This happens only for $x = 1$ and $y = 0$. At that point, we have

$$\begin{aligned} f_{xx}(1, 0) &= 2 \\ f_{xy}(1, 0) &= 0 \\ f_{yy}(1, 0) &= 2 \end{aligned}$$

Since $f_{xx}f_{yy} - f_{xy}^2 = 4$, which is positive, we conclude that the point is a minimum.

b) *Find the absolute maximum and minimum of* $f(x, y)$ *on or inside a circle of radius* 2 *centered at the origin.*

After having solved this, I realized it's probably too much of a mess to appear on the exam. The method below is what you should do if faced with a problem like this, but don't worry about being able to do one quite this messy. There are better examples in the older worksheets.

We already found the critical points in part (a), and so we only need to worry about the max/min on the boundary of the region. To get at that, the best way is to parametrize it:

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$$

Here $0 \le t \le 2\pi$. Plugging this in to the function, we get

$$\begin{aligned} g(t) = f(\mathbf{r}(t)) &= (2\cos t)^2 - 2(2\cos t) + (2\sin t)^2 \\ &= -4\cos t + 4\cos^2 t + 4\sin^2 t = 4 - 4\cos t. \end{aligned}$$

We want to know what t makes this have a max or a min. The way to do that is to set the derivative equal to 0 and see what the critical points are, Math 180-style.

$$g'(t) = 4\sin t$$

This is 0 when either $t = 0$ or $t = \pi$. We also need to consider the endpoints $t = 0$ and $t = 2\pi$. The corresponding (x, y) points are

t	$(x, y) = \mathbf{r}(t)$
0	$(2, 0)$
π	$(-2, 0)$
2π	$(2, 0)$

Now what we want to do is make a list of all the critical points on the inside of the region and on the boundary, plug in values, and see what gives the actual max and min.

(x, y)	$f(x, y)$
$(1, 0)$	-1
$(2, 0)$	0
$(-2, 0)$	8

We see that the max is 8 at $(-2, 0)$, and the min is -1 at $(1, 0)$.

Problem 8. *Consider the cardioid $r = 1 + \cos\theta$. Use an integral in polar coordinates to compute the area of the portion of the cardioid that lies to the right of the y-axis.*

The only trick here is to figure out the bounds we want on θ: in this case it isn't from 0 to 2π, but rather from $-\pi/2$ to $\pi/2$. The area is then given by the integral of the function $1\,dA$, where $dA = r\,dr\,d\theta$ in polar. This gives us

$$A = \int_{-\pi/2}^{\pi/2} \int_0^{1+\cos\theta} 1\,r\,dr\,d\theta.$$

Inner:

$$\int_0^{1+\cos\theta} r\,dr = \left.\frac{r^2}{2}\right|_0^{1+\cos\theta} = \frac{1}{2}(1+\cos\theta)^2$$

Outer:

$$\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos\theta)^2\,d\theta = 2 + \frac{3\pi}{4}.$$

(Actually doing the outer integral requires some trig identities; I doubt we'll have one like that on the exam, though you might want to be ready for it.)

Problem 9. *Consider the region bounded by the planes $x = 0$, $x = 2$, $y = 1$, $y = 2$, and $z - 4x = 5$. Use a triple integral to compute the valume of the region.*

The volume is going to be given by

$$\int_{y=1}^{2}\int_{x=0}^{2}\int_{z=0}^{5+4x} 1\,dz\,dx\,dy.$$

Inner:

$$\int_{z=0}^{5+4x} 1\,dz = 5+4x$$

Middle:

$$\int_{x=0}^{2} 5+4x\,dx = 5x+2x^2\Big|_0^2 = 18$$

Outer:

$$\int_{y=1}^{2} 18\,dy = 18.$$

That's our answer.