Math 210 (Lesieutre)
13.5: Triple integrals in spherical coordinates
March 17, 2017

Problem 1. a) *Find the spherical coordinates for the point* $(x, y, z) = (0, 1, 1)$.
The angle with the z axis is $\pi/4$, so $\phi = \pi/4$. The length is $\rho = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$. At last, $\theta = \pi/2$. So the point is $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, \pi/2)$.

b) *Find the rectangular coordinates for the point* $(1, \pi/4, \pi/4)$.
We get

$$x = \rho \sin\phi \cos\theta = (1)(\sqrt{2}/2)(\sqrt{2}/2) = 1/2,$$
$$y = \rho \sin\phi \sin\theta = (1)(\sqrt{2}/2)(\sqrt{2}/2) = 1/2,$$
$$z = \rho \cos\phi = (1)(\sqrt{2}/2) = \sqrt{2}/2,$$

and so the point is $(1/2, 1/2, \sqrt{2}/2)$.

Problem 2. *Use a triple integral to compute the volume of the unit sphere.*

To find the area of a region, you integrate the function 1. Ditto to find the volume of a region in 3D: we need to integrate the function 1 over this region. We are going to need $0 \le \rho \le 1$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$. The integral then becomes

$$\begin{aligned} V &= \int_0^1 \int_0^\pi \int_0^{2\pi} 1\, dV \\ &= \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho. \end{aligned}$$

Inner:

$$\int_0^{2\pi} \rho^2 \sin\phi \, d\theta = 2\pi\rho^2 \sin\phi.$$

Middle:

$$\int_0^\pi 2\pi\rho^2 \sin\phi \, d\phi = -2\pi\rho^2 \cos\phi \Big|_0^\pi = 4\pi\rho^2.$$

Outer:

$$\int_0^1 4\pi\rho^2 \, d\rho = 4\pi \frac{\rho^3}{3} \Big|_0^1 = \frac{4\pi}{3}.$$

This matches up with that mysterious formula $V = \frac{4}{3}\pi R^3$ that you've probably heard before.

Problem 3. *You want to integrate the function* $x^2 + y^2 + z^2$ *over the portion of the earth with latitude greater than* $45°$ *N. Convert this to an integral in spherical coordinates. (Assume the radius of the earth is* 4000*.)*

•What are the bounds? It's $0 \leq \rho \leq 4000$, $0 \leq \phi \leq \pi/4$, and $0 \leq \theta \leq 2\pi$.

•What's the function? $x^2 + y^2 + z^2 = \rho^2$.

•What's dV? In spherical, $dV = \rho^2 \sin\phi d\rho\, d\phi\, d\theta$

Thus the integral is

$$\int_0^{4000} \int_0^{\pi/4} \int_0^{2\pi} (\rho^2)\rho^2 \sin\phi d\theta\, d\phi\, d\rho = \int_0^{4000} \int_0^{\pi/4} \int_0^{2\pi} \rho^4 \sin\phi\, d\theta\, d\phi\, d\rho.$$

The problem doesn't ask us to evaluate it, so I'm not going to.

Problem 4. a) *Use a triple integral to compute the volume of the top half of the unit hemisphere.*

This is practically identical to the first problem, with the exception that the bounds on ϕ are now $0 \leq \phi \leq \pi/2$.

$$\begin{aligned} V &= \int_0^1 \int_0^{\pi=2} \int_0^{2\pi} 1\, dV \\ &= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho. \end{aligned}$$

Now take a deep breath, and evaluate the integrals one at a time.

Inner:

$$\int_0^{2\pi} \rho^2 \sin\phi\, d\theta = 2\pi\rho^2 \sin\phi.$$

Middle:

$$\int_0^{\pi/2} 2\pi\rho^2 \sin\phi\, d\phi = -2\pi\rho^2 \cos\phi\Big|_0^{\pi/2} = 2\pi\rho^2.$$

Outer:

$$\int_0^1 2\pi\rho^2\, d\rho = 2\pi\frac{\rho^3}{3}\Big|_0^1 = \frac{2\pi}{3}.$$

b) *Find the integral of the function z over the top half of the hemisphere.*

The intgeral we want is

$$\begin{aligned} &\int_0^1 \int_0^{\pi/2} \int_0^{2\pi} z\, dV \\ &= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (\rho\cos\phi)\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho \\ &= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{2}\rho^3 \sin(2\phi)\, d\theta\, d\phi\, d\rho. \end{aligned}$$

Here I used the trig identity $\cos\phi \sin\phi = \frac{1}{2}\sin(2\phi)$.

Ouch.

Inner:

$$\int_0^{2\pi} \frac{1}{2}\rho^3 \sin(2\phi)\, d\theta = \pi\rho^3 \sin(2\phi).$$

Middle:

$$\int_0^{\pi/2} \pi\rho^3 \sin(2\phi)\, d\phi = \pi\rho^3$$

Outer:

$$\int_0^{1} \pi\rho^3\, d\rho = \pi \frac{\rho^4}{4}\Big|_0^1 = \frac{\pi}{4}$$

c) *To find the z-coordinate of the center of mass of a region in 3D, you can use the formula*

$$z_{cm} = \frac{\iiint_R z\, dV}{\iiint_R 1\, dV}.$$

What is the center of mass of the northern hemisphere of the unit sphere?

Putting together our answers from (a) and (b), we get

$$z_{cm} = \frac{\pi/4}{2\pi/3} = \frac{3}{8}.$$

By symmetry the x and y coordinates are both 0. Hence the center of mass is $(0, 0, 3/8)$. Seems plausible: a hemisphere is a little bit bottom-heavy, so the center of mass should be below the midpoint.