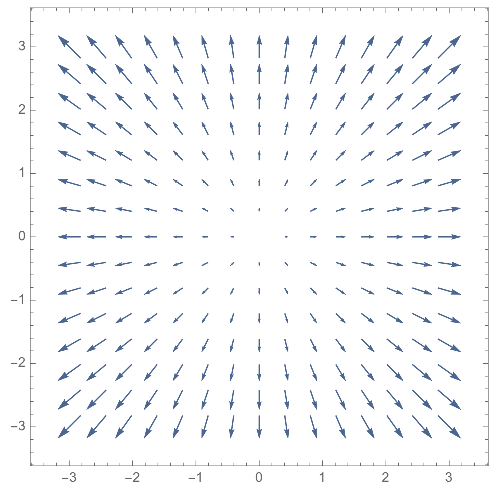
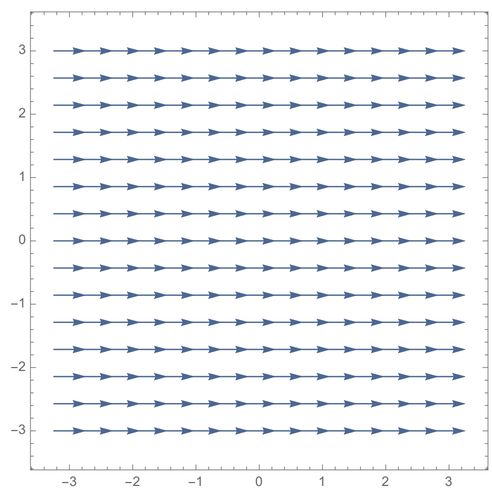
Math 210 (Lesieutre)
14.1: Vector fields
March 29, 2017

Problem 1. *Sketch each of the following vector fields.*

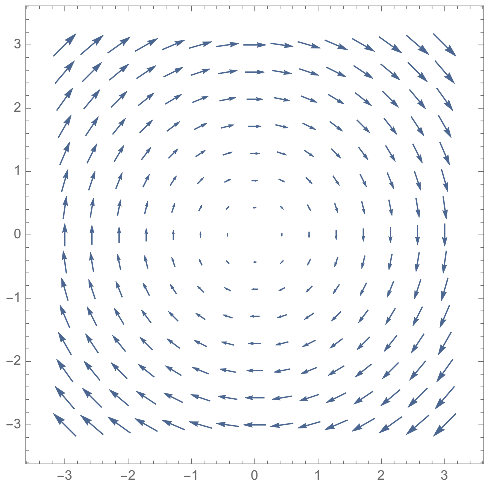
a) $\mathbf{F}_1(x, y) = \langle x, y \rangle$



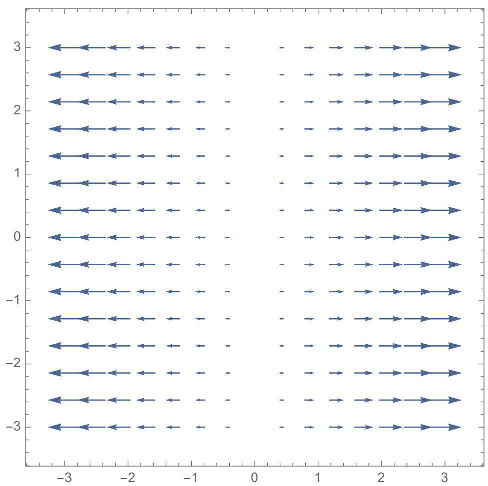
b) $\mathbf{F}_2(x, y) = \langle 3, 0 \rangle$



c) $\mathbf{F}_3(x, y) = y\mathbf{i} - x\mathbf{j}$



d) $\mathbf{F}_4(x, y) =$ *student's choice (make one up, not constant)*
I made up the vector field $\mathbf{F}_5(x, y) = \langle x, 0\rangle$. Here's a plot.



(You probably made up something different, so hopefully your plot doesn't look the same.)

Problem 2. *Write down formulas for the vector fields described.*

a) *A field* $\mathbf{F}$ *which always points clockwise, and has length* 1 *for any* x *and* y*.*
To do this, we can start with the field from (c) of the previous problem. This already points in the right direction, so all we need to do to get the answer we want is adjust it so that the length is always 1. To do that, just divide by the length:

$$\mathbf{F} = \left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle.$$

b) *A 3D vector field which points directly in towards the origin, with length proportional to* $1/r^2$ *(this could be a gravitational field).*

The field $\langle -x, -y, -z \rangle$ points in the direction we want, but the length is r. We need to divide by r^3 to get the length to be $1/r^2$, and so our field is going to be

$$\left\langle -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

Problem 3. *Suppose that a mass rests at* $(0,0)$*. The gravitational potential due to the mass at a point* (x,y) *is given by* $f(x,y) = \frac{1}{r}$*, where* r *is the distance from* (x,y) *to* $(0,0)$*.*

a) *Compute the gradient field associated to this potential function.*
It's just given by the gradient, which we compute in the usual way. The function is

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

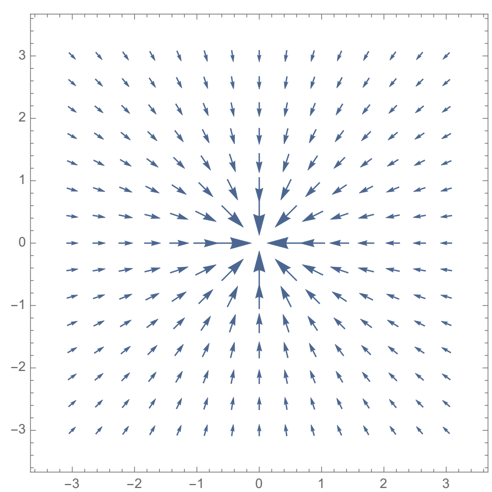
Notice that

$$\begin{aligned} \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2}} &= \frac{\partial}{\partial x}(x^2+y^2)^{-1/2} = -\frac{1}{2}(x^2+y^2)^{-3/2}(2x) \\ &= -x(x^2+y^2)^{-3/2} = -\frac{x}{(x^2+y^2)^{3/2}}. \end{aligned}$$

So the gradient is

$$\nabla f = \left\langle -\frac{x}{(x^2+y^2)^{3/2}}, -\frac{y}{(x^2+y^2)^{3/2}} \right\rangle$$

b) *Try to sketch your vector field. Do the vectors get shorter or longer the further we get from the origin?*
Here's the sketch:



They get shorter. In fact, the length of the vector at position (x,y) is given by $\frac{1}{x^2+y^2} = \frac{1}{r^2}$. This is what you know from Newton's law of universal gravitation: the field is proportional to $1/r^2$ (really it goes the other way: we found the formula for the potential assuming we already knew what the field looks like).

Problem 4. *Compute and sketch the gradient field associated with the function $f(x, y) = \tan^{-1}(y/x)$.*

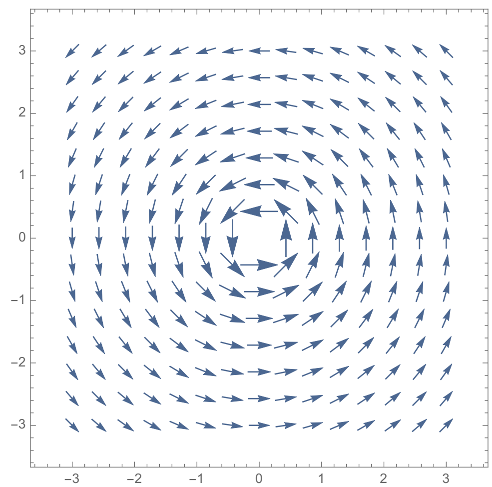
First let's compute the derivatives. Remember that the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$. We have

$$\frac{\partial}{\partial x}\tan^{-1}(y/x) = \frac{1}{(y/x)^2+1}\left(-\frac{y}{x^2}\right) = -\frac{x^2}{y^2+x^2}\frac{y}{x^2} = -\frac{y}{x^2+y^2},$$
$$\frac{\partial}{\partial y}\tan^{-1}(y/x) = \frac{1}{(y/x)^2+1}\left(\frac{1}{x}\right) = \frac{x^2}{y^2+x^2}\frac{1}{x} = \frac{x}{x^2+y^2}.$$

So the gradient field is

$$\mathbf{F} = \nabla f = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

This is another radial field, with the vectors getting shorter as we get further from the origin.



Problem 5. *Consider the two fields*

$$\mathbf{F}_1(x, y) = \left\langle x^2, y^2 \right\rangle, \quad \mathbf{F}_2(x, y) = \left\langle y^2, x^2 \right\rangle.$$

One of these is the gradient field of some function, and the other one isn't. Which is which? How can you tell?

Let's try to find a potential function. If $f(x, y)$ is a potential for the first function, then $f_x = x^2$, and so $f = \frac{x^3}{3} + g(y)$ (here $g(y)$ could be any function, but it only depends on y, which makes its x partial 0). Then we want $f_y = y^2$, and we know $f_y = g_y$, so $g_y = y^2$ and $g(y) = y^3/3$. So we found a potential: $f(x, y) = \frac{x^3}{3} + \frac{y^3}{3}$.

How about the second one? Well, $f_x = y^2$, so $f(x, y) = xy^2 + g(y)$. This gives $f_y = 2xy + g_y$, which we want to be y^2. No function g_y is going to make this work, so this one isn't a gradient.

(More on this later.)