Math 210 (Lesieutre)
14.2: Line integrals
March 31, 2017

Problem 1. *Consider the scalar function* $f(x, y) = xy$. *Compute* $\int_C f\,ds$ *for the listed paths.*

a) *Let* C *be a straight line path from* $(2, 0)$ *to* $(0, 2)$.
First we parametrize. Using our usual formula for a straight line parametrization, this comes out to

$$\mathbf{r}(t) = \langle 2, 0\rangle + t\,\langle -2, 2\rangle = \langle 2 - 2t, 2t\rangle\,.$$

We're also going to need to know

$$\mathbf{r}'(t) = \langle -2, 2\rangle$$
$$|\mathbf{r}'(t)| = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}.$$

(In this case it doesn't depend on t. Sometimes it will, when the speed of the particle is varying.)
So $x(t) = 2 - 2t$, $y(t) = 2t$. The integral is now

$$\int_C f\,ds = \int_0^1 (2 - 2t)(2t)(\sqrt{2})\,dt = 4\sqrt{2}\int_0^1 t - t^2\,dt = \frac{2\sqrt{2}}{3}.$$

b) *Let* C *be the part of a circle of radius* 2 *centered at the origin that lies in the first quadrant, oriented counterclockwise.*
We follow the same strategy, starting with a parametrization. In this case we want a circular path, and so

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$$

with the range $0 \le t \le \pi/2$. We're also going to need to know

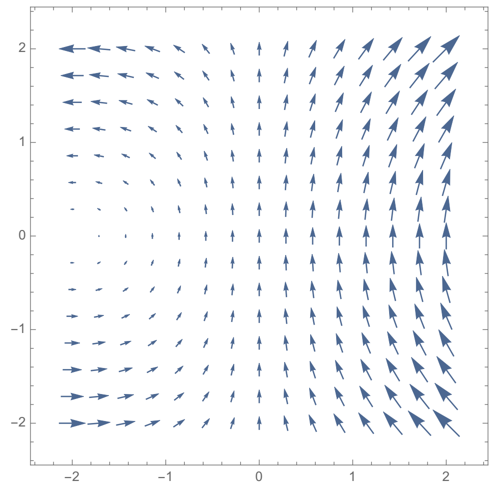
$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$$
$$|\mathbf{r}'(t)| = \sqrt{(-2\sin t)^2 + (2cost)^2} = \sqrt{4} = 2.$$

(Again, there's no t dependence, because our particle is moving at constant speed.)
So $x(t) = 2\cos t$, $y(t) = 2\sin t$. The integral is now

$$\int_C f\,ds = \int_0^1 (2\cos t)(2\sin t)(2)\,dt = 8\int_0^{\pi/2} \cos t\sin t\,dt = 8\frac{1}{2} = 4.$$

c) *What is the average value of* $f(x, y)$ *along the path from (b)?*
We just need to take our answer and divide it by the length of the path. It's a quarter circle, so the length is $\frac{1}{4}(4\pi) = \pi$. The average value is then going to be $\frac{4}{\pi}$.

Problem 2. *Now consider the vector field* $\mathbf{F}(x, y) = \langle xy, 1 + x \rangle$*:*



Let C_1 *be the quarter-circle path from problem 1b, and* C_2 *the straight line path from* $(0, 0)$ *to* $(2, 0)$*.*

a) *Do you expect* $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ *to be positive, negative, or zero? How about* $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$*?*
Along the path C_1, we have $y = 0$, and so the vector field points directly upward. This means that it's perpendicular to our path, and so not doing any work: this wind is coming straight from the side, and so neither helps nor hinders our walking. Thus we expect that the integral is 0.

For the second path, looking at the picture, it appears that the wind is more or less at our backs, at least until towards the end of the path. So it should help, and we expect that the integral is positive.

b) *Compute* $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$*.*
The parametrization is $\mathbf{r}(t) = \langle 2t, 0 \rangle$, so that $x(t) = 2t$ and $y(t) = 0$. Plugging in, we get

$$\mathbf{F}(t) = \langle (2t)(0), 1 + (2t) \rangle$$

We have $\mathbf{r}'(t) = \langle 2, 0 \rangle$. So the integral is

$$\int_{t=0}^{1} \langle 0, 1 + 2t \rangle \cdot \langle 2, 0 \rangle \, dt = \int_{0}^{1} 0 \, dt = 0.$$

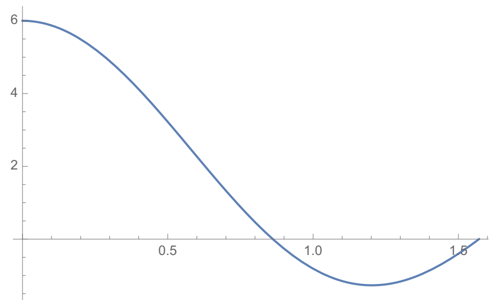
c) *Compute* $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$*, or at least set up the corresponding single variable integral.*
This is a little more work. We have $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$. Then

$$\begin{aligned} \mathbf{F}(t) &= \langle xy, 1 + x \rangle = \langle (2\cos t)(2\sin t), 2\cos t + 1 \rangle \\ &= \langle 4\cos t \sin t, 2\cos t + 1 \rangle \\ \mathbf{r}'(t) &= \langle -2\sin t, 2\cos 2 \rangle \end{aligned}$$

What we want is then

$$\int_0^{\pi/2} \langle 4\cos t \sin t, 2\cos t + 1\rangle \cdot \langle -2\sin t, 2\cos t\rangle \; dt$$
$$= \int_0^{\pi/2} \left(-8\cos t \sin^2 t + 4\cos^2 t + 2\cos t\right) \; dt = \pi - \frac{2}{3}.$$

It takes a bit of trig to integrate this mess. Here's a plot of the function we're integrating:



It's mostly positive, which means that the wind is more or less at our backs, and so doing positive work. So it should come as no surprise that the integral is positive.

Problem 3. *Consider the change of variables T given by $u = 2x + y$ and $v = x + y$. Let R be the parallelogram in the xy-plane with vertices at $(0, 0)$, $(-1, 2)$, $(3, -3)$, and $(2, -1)$. Compute*

$$\iint_R xy \, dx \, dy.$$

First we need to express the region in uv-coordinates. What are these points? Using the formulas for u and v, we find that they are respectively $(u, v) = (0, 0)$, $(0, 1)$, $(3, 0)$, and $(3, 1)$. This is a coordinate rectangle!

We need x and y as a function of u and v, instead of the other way. Notice that $u - v = (2x + y) - (x + y) = x$. Then $y = v - x = v - (u - v) = 2v - u$. So $x(u, v) = u - v$ and $y(u, v) = 2v - u$. (There are other ways to do the algebra to get this answer – it's up to you.)

Now, the function xy becomes $(u - v)(2v - u)$.

At last, what to make of $dx\,dy$. We need the Jacobian, which is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1.$$

That means we get off easy this time: $dx\,dy = J(u, v)\,du\,dv = 1\,du\,dv = du\,dv$.

So our integral is

$$\int_{v=0}^{1} \int_{u=0}^{3} (u - v)(2v - u)\,du\,dv = \cdots = -\frac{17}{4}.$$