Math 210 (Lesieutre)
14.3: Conservative vector fields, II
April 5, 2017

Problem 1. *Consider the field* $\mathbf{F}_1 = \langle x^2, y^2 \rangle$. *Last time we saw that this is a conservative field, and that* $\phi(x,y) = \frac{x^3}{3} + \frac{y^3}{3}$ *is a potential function.*

a) *Compute the line integral of* $\mathbf{F}$ *along a straight line path from* $(0,0)$ *to* $(1,1)$.
Let's do the first one first. We have $\mathbf{r}(t) = \langle t, t \rangle$ with $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 1, 1 \rangle$. Plugging in $x(t) = t$ and $y(t) = t$, our integral becomes

$$\int_0^1 \left\langle t^2, t^2 \right\rangle \cdot \langle 1, 1 \rangle \, dt = \int_0^1 2t^2 \, dt = \frac{2}{3}.$$

b) *Compute the line integral of* $\mathbf{F}$ *along a path that goes from* $(0,0)$ *to* $(1,0)$ *and then to* $(1,1)$. *What do you notice?*
We have to integrate over the two parts separately, then add the results together. For the first piece, $\mathbf{r}(t) = \langle t, 0 \rangle$ with $0 \le t \le 1$. This gives us $\mathbf{r}'(t) = \langle 1, 0 \rangle$, and we have to integrate

$$\int_0^1 \left\langle t^2, 0 \right\rangle \cdot \langle 1, 0 \rangle \, dt = \int_0^1 t^2 \, dt = \frac{1}{3}.$$

For the second piece, $\mathbf{r}(t) = \langle 0, t \rangle$ and $\mathbf{r}'(t) = \langle 0, 1 \rangle$. The corresponding integral is

$$\int_0^1 \left\langle 0, t^2 \right\rangle \cdot 0, 1 \, dt = \int_0^1 t^2 \, dt = \frac{1}{3}.$$

The total integral along this bent path is $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.
What do I notice? Well, we got the same answer for both paths. This is suspicious.

c) *Repeat the previous two calculations using the fundamental theorem for line integrals.*
Both paths start at $(0,0)$ and end at $(1,1)$. We know that a potential function is

$$\phi(x,y) = \frac{x^3}{3} + \frac{y^3}{3},$$

so the integral is just

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(1,1) - \phi(0,0)$$

Problem 2. *Consider the vector field* $\mathbf{F}(x,y) = \langle y+1, x+1 \rangle$.

a) *Verify that* $\mathbf{F}$ *is a conservative field and find a potential function* $\phi(x,y)$.
We have $f(x,y) = y+1$ and $g(x,y) = x+1$. Then $f_y = 1$ and $g_x = 1$, and because the vector field is defined for all x and y that means it's conservative.
We need $\phi_x = y+1$, so $\phi = xy + x + c(y)$. Then $\phi_y = x + c_y$, which is supposed to equal $x+1$. So $c_y(y) = 1$, whence $c(y) = y$. Our potential function is then $\phi(x,y) = xy + x + y$.

b) *Let C be a semicircular path from $(1,0)$ to $(-1,0)$. Use the fundamental theorem for line integrals to compute* $\int_C \mathbf{F} \cdot d\mathbf{r}$.

The path starts at $(1,0)$ and ends at $(-1,0)$. That means that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(-1,0) - \phi(1,0) = -1 - 1 = -2.$$

c) *Let C be a path that goes the whole way around the unit circle. Check by a direct computation that*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

We parametrize the path by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$, with bounds $0 \le t \le 2\pi$. Now we plug this in to $\mathbf{F} = \langle y+1, x+1 \rangle$ and integrate.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle \sin t + 1, \cos t + 1 \rangle \cdot -\sin t, \cos t \, dt \\
&= \int_0^{2\pi} \left(-\sin^2 t - \sin t + \cos^2 t + \cos t \right) \, dt \\
&= \int_0^{2\pi} \cos(2t) - \sin t + \cos t \, dt \\
&= \frac{1}{2} \sin(2t) + \cos t + \sin t \Big|_0^{2\pi} = 0.
\end{aligned}$$

Problem 3. *Let $\mathbf{F}$ be the vector field $\langle x, y \rangle$ and let C be a straight line from $(1,1)$ to $(-1,1)$.*

a) *Do you expect the flux of $\mathbf{F}$ across C to be positive, negative, or zero?*

The field is flowing outward from the origin, across C. We expect the flux to be positive.

b) *Compute the flux* $\int_C \mathbf{F} \cdot \mathbf{n} \, ds$.

The path is parametrized by $\mathbf{r}(t) = \langle 1 - 2t, 1 \rangle$. The tangent vector is $\mathbf{r}'(t) = \langle -2, 0 \rangle$. The normal is $\mathbf{n} = \langle 0, 2 \rangle$ (switch the two parts of $\mathbf{r}'$ and then multiply the second by -1). This is an upward normal vector.

Our integral is then

$$\int_0^1 \langle 1 - 2t, 1 \rangle \cdot 0, 2 \, dt = \int_0^1 2 \, dt = 2.$$

Positive, like we thought.