Math 210 (Lesieutre)
14.4: Green's theorem, I
April 7, 2017

Circulation form:	Flux form:
$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \operatorname{curl}\mathbf{F}\,dA$	$\oint_C \mathbf{F}\cdot \mathbf{n}\,ds = \iint_R \operatorname{div}\mathbf{F}\,dA$
$\operatorname{curl}\mathbf{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$	$\operatorname{div}\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$

Problem 1. *Compute* $\operatorname{div}\mathbf{F}$ *and* $\operatorname{curl}\mathbf{F}$ *for the indicated vector fields.*

a) $\mathbf{F}_1(x,y) = \langle x, y\rangle$

Here $f(x,y) = x$ and $g(x,y) = y$. The divergence is $f_x + g_y = 1 + 1 = 2$. This seems physically reasonable: the field flows outward everywhere.

The curl is $g_x - f_y = 0 - 0 = 0$. Also sensible: from a sketch of the vector field, it looks like a small propellor wouldn't turn.

b) $\mathbf{F}_2(x,y) = \langle -y, x\rangle$

This time we get $\operatorname{div}\mathbf{F} = 0 + 0 = 0$ while $\operatorname{curl}\mathbf{F} = 1 - (-1) = 2$. This also makes sense: this field looks like a whirlpool around the original, and a propellor stuck in anywhere would spin counterclockwise.

Problem 2. *Let* $\mathbf{F} = \langle -y, x\rangle$*, and let* C *be a path around the unit circle starting at* $(1,0)$ *and going counterclockwise. Verify the circulation form of Green's theorem by computing both sides directly.*

Let's first do the line integral. We parametrize the path as $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. Plugging this into the vector field we get $\mathbf{F} = \langle -y, x\rangle = \langle -\sin t, \cos t\rangle$. We also need to know $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$ (yes, it's the same thing as $\mathbf{F}$; this is basically a coincidence since I wanted an easy integral). We then compute

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\; dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

Now let's do the area integral. The curl is $\operatorname{curl}\mathbf{F} = g_x - f_y = 1 - (-1) = 2$, and so we want:

$$\iint_R \operatorname{curl}\mathbf{F}\,dA = \iint_R 2\,dA = 2(\text{area of } R) = 2\pi.$$

Seems to work!

Problem 3. *Let R be the region bounded by $y = 1 - x^2$ and $y = 0$, and let C be a path that goes around the region counterclockwise, starting at $(1, 0)$. Verify the flux form of Green's theorem for the vector field $\mathbf{F} = \langle x + y, xy \rangle$.*

Start with the double integral this time. The divergence is $f_x + g_y = 1 + x$, and so we want to integrate the function $1 + x$ over the triangle. The top edge of the region $y = 1 - x^2$, so the integral we want is

$$\int_{x=-1}^{1} \int_{y=0}^{1-x^2} 1 + x \, dy \, dx = \int_{x=-1}^{1} (1 - x^2)(1 + x) \, dx = \int_{x=-1}^{1} 1 + x - x^2 - x^3$$
$$= \left(x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_{-1}^{1} = \frac{11}{12} - \left(-\frac{5}{12} \right) = \frac{16}{12} = \frac{4}{3}.$$

Now we need to compute the flux across the edges, of which there are two. First, let's do the top, curved edge; call it C_1

$$\mathbf{r}(t) = \left\langle -t, (-t)^2 \right\rangle = \left\langle -t, 1 - t^2 \right\rangle$$
$$\mathbf{r}'(t) = \langle -1, -2t \rangle$$
$$\mathbf{n}(t) = \langle -2t, 1 \rangle .$$

The range is $-1 \leq t \leq 1$. Plugging in, we have

$$\mathbf{F} = \langle x + y, xy \rangle = \left\langle -t + (1 - t^2), -t + t^3 \right\rangle = \left\langle 1 - t - t^2, t^3 - t \right\rangle .$$

This normal vector points upward, which is out of the region; that's as it should be, so everything looks good.

The flux is

$$\int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{t=-1}^{1} \left\langle 1 - t - t^2, t^3 - t \right\rangle \cdot \langle -2t, 1 \rangle \, dt$$
$$= \int_{t=-1}^{1} \left(-2t + 2t^2 + 2t^3 \right) + \left(t^3 - t \right) \, dt$$
$$= \int_{t=-1}^{1} 3t^3 + 2t^2 - 3t \, dt = \frac{4}{3}.$$

Now we have to compute the flux across the bottom edge. This is a straight line from $(-1, 0)$ to $(1, 0)$, and so we get

$$\mathbf{r}(t) = \langle 2t, 0 \rangle$$
$$\mathbf{r}'(t) = \langle 2, 0 \rangle$$
$$\mathbf{n}(t) = \langle 0, -2 \rangle .$$

Our range this time is $-1 \leq t \leq 1$. Plugging in we have $\mathbf{F} = \langle x+y, xy \rangle = \langle 2t, 0 \rangle$, and the integral is

$$\int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t=0}^{1} \langle 2t, 0 \rangle \cdot \langle 0, -2 \rangle = \int_{t=0}^{1} 0\, dt = 0.$$

The total flux is then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = \frac{4}{3} + 0 = \frac{4}{3}.$$

This confirms that

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA.$$

Problem 4. *What does the circulation form of Green's theorem tell us when* $\mathbf{F}$ *is a conservative vector field?*

If $\mathbf{F}$ is conservative, then $f_y = g_x$. This means that $\operatorname{curl} \mathbf{F} = g_x - f_y = 0$. The theorem says that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F}\, dA$$

The right-hand side is 0, and so the theorem is telling us that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Of course, we already knew that because of the fundamental theorem for line integrals!