Math 210 (Lesieutre)
14.4: Green's theorem, II
April 10, 2017

Circulation form:	Flux form:
$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F}\, dA$	$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$
$\operatorname{curl} \mathbf{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$	$\operatorname{div} \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$

Problem 1. *Let $\mathbf{F} = \langle -y, x \rangle$, and let C be a path around the unit circle starting at $(1, 0)$ and going counterclockwise. Verify the circulation form of Green's theorem by computing both sides directly.*

Let's first do the line integral. We parametrize the path as $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Plugging this into the vector field we get $\mathbf{F} = \langle -y, x \rangle = \langle -\sin t, \cos t \rangle$. We also need to know $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ (yes, it's the same thing as $\mathbf{F}$; this is basically a coincidence since I wanted an easy integral). We then compute

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

Now let's do the area integral. The curl is $\operatorname{curl} \mathbf{F} = g_x - f_y = 1 - (-1) = 2$, and so we want:

$$\iint_R \operatorname{curl} \mathbf{F}\, dA = \iint_R 2\, dA = 2(\text{area of } R) = 2\pi.$$

Seems to work!

Problem 2. *Suppose that you have a region R bounded by a curve C. What does the circulation form of Green's theorem tell you when it's applied to the field $\mathbf{F} = \langle 0, x \rangle$?*

Well, the curl of this field is

$$\operatorname{curl} \mathbf{F} = g_x - f_y = 1.$$

So Green's theorem says

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_R 1\, dA = \text{area}(R).$$

The line integral is the area!

There is actually a gadget called a planimeter that you can fix to a table and then track out a curve with a pencil. The device is geared so that as you trace it out, it adds up $\oint \mathbf{F} \cdot d\mathbf{r}$. Once you trace around a complete loop, it computes the area of the region. This is without using any computers! You can find some description of how these things work if you google around a little bit.

Problem 3. *Use Green's theorem to compute the outward flux of the field* $\mathbf{F} = \langle x^2 y, y\rangle$ *across a counterclockwise semicircular path from* $(2,0)$ *to* $(0,-2)$.

One could probably do this by a direct integral, but there is a useful trick with Green's theorem to avoid it.

Let C be a closed loop that starts at $(2,0)$, follows the path in question to $(-2,0)$ around the top of the circle, and then goes in a straight line from $(-2,0)$ to $(2,0)$. Let's give names to the two parts of this path: C_1 is the curved part around the top, and C_2 is the straight part around the bottom. Let R be the interior of C, which is a half disk.

Then Green's theorem tells us that

$$\iint_R \operatorname{div}\mathbf{F}\, dA = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

If we could compute $\iint_R \operatorname{div}\mathbf{F}\, dA$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, that would give us $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, which is a much worse integral than either of the other two.

Let's start with the double integral. The divergence is

$$\operatorname{div}\mathbf{F} = f_x + g_y = 2xy + 1.$$

So we want to take the double integral of that over the region R. Remember that we're integrating over a half-disk, so life is going to be a lot more pleasant if we do this integral in polar coordinates. The function is

$$\begin{aligned} f(x,y) &= 2xy + 1 = 2(r\cos\theta)(r\sin\theta) + 1 \\ &= 2r^2\cos\theta\sin\theta + 1 \\ &= r^2 \sin 2\theta + 1. \end{aligned}$$

Don't forget to use $r\,dr\,d\theta$ for dA, since we're working in polar.

Our integral is now:

$$\begin{aligned} \iint_R \operatorname{div}\mathbf{F}\, dA &= \int_{r=0}^{2}\int_{\theta=0}^{\pi} \big(r^2\sin(2\theta) + 1\big)\, r\, d\theta\, dr \\ &= \int_{r=0}^{2}\int_{\theta=0}^{\pi} r^3 \sin(2\theta) + r\, d\theta\, dr \end{aligned}$$

The inner one is

$$\begin{aligned} \int_{\theta=0}^{\pi} r^3 \sin 2\theta + r\, d\theta &= r^3 \int_{\theta=0}^{\pi} \sin 2\theta\, d\theta + \int_{\theta=0}^{\pi} r\, d\theta \\ &= r^3 \left.\frac{-\cos 2\theta}{2}\right|_0^{\pi} + \pi r = \pi r. \end{aligned}$$

The outer is

$$\int_{r=0}^{2} \pi r \, dr = \left. \frac{\pi r^2}{2} \right|_0^2 = 2\pi.$$

We also need to compute

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

This is just an old-fashioned circulation integral. The parametrization is

$$\begin{aligned} \mathbf{r}(t) &= \langle -2, 0 \rangle + t \langle 4, 0 \rangle = \langle -2 + 4t, 0 \rangle \\ \frac{d\mathbf{r}(t)}{dt} &= \langle 4, 0 \rangle . \end{aligned}$$

with $0 \le t \le 1$.

Plugging that in,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{1} \left\langle (-2+4t)^2(0), 0 \right\rangle \cdot \langle 4, 0 \rangle \; dt = \int_{t=0}^{1} 0 \, dt = 0.$$

Going back to our original equation,

$$\begin{aligned} \iint_R \operatorname{div} \mathbf{F} \, dA &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ 2\pi &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + 0 \\ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= 2\pi, \end{aligned}$$

which is what we were trying to compute. In this case the original integral wouldn't have been all that bad, but there are situations where this trick is really very handy.

Problem 4. *Evaluate the line integral* $\oint_C (2x+y)\,dx + (x+y)\,dy$*, where* C *is a square with opposite corners at* $(0,0)$ *and* $(1,1)$*.*

First, what is this weird notation? I haven't used it very heavily.

Think of it as

$$\oint_C \langle x+y, 2x+y \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \, dt$$

So this is a circulation integral. (Sometimes you'll see $\int_C f\,dy - g\,dx$; this is $\langle f, g \rangle \cdot dy, -dx$, which is the flux.)

Anyway, nobody wants to do four separate line integrals, so we'll use Green's theorem instead. The field in question in $\mathbf{F} = \langle x + y, 2x + y \rangle$, and so $\operatorname{curl} \mathbf{F} = g_x - f_y = 2 - 1 = 1$. Then Green's theorem tells us that

$$\oint_C (2x + y)\, dx + (x + y)\, dy = \iint_R 1\, dA = 1\, \mathrm{area}(R) = 1,$$

since the square has area 1.

Problem 5. *Let R be the unit circle. Convert the double integral $\iint_R xy + \sin x\, dA$ into a line integral; you do not need to evaluate.*

There's more than one way to do this. If we can find a field $\mathbf{F}$ for which $\operatorname{div} \mathbf{F} = 2xy + \sin x$, we'd be done, because we could then use Green's theorem. Here are a couple fields that would work as such an $\mathbf{F}$:

$$\mathbf{F}_1(x, y) = \left\langle x^2 y - \cos x, 0 \right\rangle,$$
$$\mathbf{F}_2(x, y) = \left\langle x^2 y, y \sin x \right\rangle$$

then

$$\iint_R xy + \sin x\, dA = \oint_C \left\langle x^2 y, y \sin x \right\rangle \cdot d\mathbf{r},$$

according to Green's theorem. We could then evaluate this integral.

By finding a field $\mathbf{F}$ with $\operatorname{curl} \mathbf{F} = 2xy + \sin x$, we could accomplish the same thing using the other version of Green's theorem.