Math 210 (Lesieutre)
14.5: Divergence and curl
April 12, 2017

Problem 1. *Let* $\mathbf{F}$ *be the vector field* $\mathbf{F} = \langle xyz, ye^x, z \rangle$.

a) *Compute the divergence* $\nabla \cdot \mathbf{F}$.
The divergence is a scalar function, given in this case by

$$\nabla \cdot \mathbf{F} = f_x + g_y + h_z = yz + e^x + 1.$$

b) *Compute the curl* $\nabla \times \mathbf{F}$.
The curl is given by

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \\
&= (0 - 0)\, \mathbf{i} + (xy - 0)\, \mathbf{j} + (ye^x - xz)\, \mathbf{k} \\
&= xy\mathbf{j} + (ye^x - xz)\mathbf{k} = \langle 0, xy, ye^x - xz \rangle .
\end{aligned}$$

c) *Compute the divergence of the curl,* $\nabla \cdot (\nabla \times \mathbf{F})$.
We found that the curl was given by

$$\mathbf{G} = \nabla \times \mathbf{F} = \langle 0, xy, ye^x - xz \rangle .$$

The divergence is then

$$\nabla \cdot \mathbf{G} = f_x + g_y + h_z = 0 + x + (0 - x) = 0.$$

In fact this is not a coincidence: the divergence of the curl is always 0.

Problem 2. *Suppose that we have a 3D vector field of the form* $\mathbf{F} = \langle f(x, y), g(x, y), 0 \rangle$ *(i.e.* f *and* g *only depend on* x *and* y*, and* $h = 0$*). What is the (3D) curl* $\nabla \times \mathbf{F}$*? What do you notice about this?*

The curl is given by

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k} \\
&= (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (g_x - f_y)\mathbf{k}.
\end{aligned}$$

The $g_x - f_y$ is the 2d curl of the corresponding 2d field $\mathbf{F} = \langle f, g \rangle$.

Problem 3. *Consider the function $\phi(x,y,z) = xy^2z$.*

a) *Compute the gradient $\mathbf{F} = \nabla\phi$. (This $\mathbf{F}$ is a conservative field.)*
The gradient is given by $\langle \phi_x, \phi_y, \phi_z \rangle = \langle y^2z, 2xyz, xy^2 \rangle$.

b) *Compute the curl of the gradient, $\nabla \times \nabla\phi$.*
Once again, the curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} \\ &= (2xy - 2xy)\mathbf{i} + (y^2 - y^2)\mathbf{j} + (2yz - 2yz)\mathbf{k}.\end{aligned}$$

In fact the curl of a gradient is always $\mathbf{0}$, which gives us a way to check remember how to check whether a 3d field is conservative.

Problem 4. *Consider the vector field $\mathbf{F} = \langle 1,0,3 \rangle \times \mathbf{r}$, where $\mathbf{r} = \langle x,y,z \rangle$. This is an example of a* rotation vector field*.*

a) *Multiply out the cross product to find a formula for $\mathbf{F}$.*
I get

$$\mathbf{F} = \langle 1,0,3 \rangle \times \langle x,y,z \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ x & y & z \end{vmatrix} = (0-3y)\mathbf{i} - (z-3x)\mathbf{j} + (y-0)\mathbf{k} = \langle -3y, 3x - z, y \rangle.$$

b) *Compute the curl of this field.*
The curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} \\ &= (1-(-1))\mathbf{i} + (0-0)\mathbf{j} + (3-(-3))\mathbf{k} = 2\mathbf{i} + 6\mathbf{k} = \langle 2,0,6 \rangle.\end{aligned}$$

It's no coincidence that this is twice the vector $\langle 1,0,3 \rangle$ that we started with – this is how things always work for rotation fields $\mathbf{F} = \mathbf{a} \times \mathbf{r}$.