Math 210 (Lesieutre)
14.6: Surface integrals
April 15, 2017

Problem 1. *Let S be the outer shell of a cylinder of height 3 and radius 4, with base centered at the origin.*

a) *Give a parametrization for S, and compute $\mathbf{t}_u \times \mathbf{t}_v$.*
How do we describe a point on a cylinder? We use cylindrical coordinates (r, θ, z). In this case, r is always 3, so we're going to use the other two. Take u to be θ and v to be z. Our coordinates are

$$\mathbf{r}(u, v) = \langle 4\cos u, 4\sin u, v \rangle .$$

Then

$$\begin{aligned} \mathbf{t}_u &= \frac{\partial \mathbf{r}}{\partial u} = \langle -4\sin u, 4\cos u, 0 \rangle \\ \mathbf{t}_v &= \frac{\partial \mathbf{r}}{\partial v} = \langle 0, 0, 1 \rangle \\ \mathbf{t}_u \times \mathbf{t}_v &= \langle 4\cos u, 4\sin u, 0 \rangle \\ |\mathbf{t}_u \times \mathbf{t}_v| &= \sqrt{(4\cos u)^2 + (4\sin u)^2 + 0^2} = \sqrt{16} = 4. \end{aligned}$$

The bounds would be $0 \leq u \leq 2\pi$ and $0 \leq v \leq 3$.

b) *How would you compute the integral $\iint_S xz\, dS$?*
The function xz becomes $(4\cos u)(v) = 4v\cos u$ in these coordinates. So we'd use the integral

$$\iint_S xz\, dS = \int_{u=0}^{2\pi} \int_{v=0}^{3} 4v\cos u\,(4\, du\, dv).$$

The extra 4 at the end comes from $|\mathbf{t}_u \times \mathbf{t}_v|\ du\, dv$.

Problem 2. *Give parametrizations for the following regions.*

a) *A sphere of radius 7.*
We're going to use the spherical coordinates θ and ϕ as our coordinates. Let's take u to be ϕ and v to be θ. We're working on the surface S, and so ρ is always going to be 7.
This gives

$$\mathbf{r}(u, v) = \langle 7\sin u\cos v, 7\sin u\sin v, 7\cos u \rangle .$$

In this case, we want $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

b) *The part of the plane $z = 5 + 2x + y$ lying above the unit square in the xy-plane.*
In this case our parameters are just going to be $u = x$ and $v = y$, and we'll use $z = 5 + 2u + v$. So we want

$$\mathbf{r}(u, v) = \langle u, v, 5 + 2u + v \rangle .$$

Our bounds are $0 \leq u \leq 1$, $0 \leq v \leq 1$.
I guess the problem doesn't ask us to compute the area element dS, but what's the harm?

$$\begin{aligned}
\mathbf{t}_u &= \frac{\partial \mathbf{r}}{\partial u} = \langle 1, 0, 2 \rangle \\
\mathbf{t}_v &= \frac{\partial \mathbf{r}}{\partial v} = \langle 0, 1, 1 \rangle \\
\mathbf{t}_u \times \mathbf{t}_v &= \langle -2, -1, 1 \rangle \\
|\mathbf{t}_u \times \mathbf{t}_v| &= \sqrt{6}.
\end{aligned}$$

Problem 3. *Consider the surface S comprising the portion of the paraboloid $z = 1 - x^2 - y^2$ lying above the unit disk.*

a) *Give a parametrization of S; be sure to include bounds on u and v.*
We're just going to take $x(u, v) = u$ and $y(u, v) = v$. Then $z(u, v) = 1 - u^2 - v^2$. We require u and v to be in the unit disc R in the uv-plane.
It's also possible to skip straight to parametrizing it using polar coordinates for the x and y coordinates, which is actually a little easier. I'm going to do it the long way, but on the homework you might consider taking this approach the entire way through. In this case, we'd take u to be r and v to be θ. Then

$$\begin{aligned}
x(u, v) &= u \cos v \\
y(u, v) &= u \sin v \\
z(u, v) &= 1 - x^2 - y^2 = 1 - v^2.
\end{aligned}$$

The bounds will just be $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$ (remember that u and v are really just the coordinates from polar).

b) *Compute the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u}$, $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v}$, the cross product $\mathbf{t}_u \times \mathbf{t}_v$, and the length $|\mathbf{t}_u \times \mathbf{t}_v|$.*
I get

$$\begin{aligned}
\mathbf{t}_u &= \langle 1, 0, -2u \rangle \\
\mathbf{t}_v &= \langle 0, 1, -2v \rangle \\
\mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle \\
|\mathbf{t}_u \times \mathbf{t}_v| &= \sqrt{(2u)^2 + (2v)^2 + 1} = \sqrt{1 + 4u^2 + 4v^2}.
\end{aligned}$$

If you were doing it the polar way, you'd instead take the derivatives of the alternate answer given in the first part of problem 1.

c) *Set up an integral for the surface area of this part of the paraboloid. If you have time, compute the integral.*

We just want to integrate the function 1, so our integral is

$$\iint_S 1\, dS = \iint_R 1\,\sqrt{1+4u^2+4v^2}\, du\, dv$$

Here R is the unit disk in the uv-plane.

To actually compute this, we have little choice but to convert the integral into polar. The integrand $\sqrt{1+4u^2+4v^2}$ is $\sqrt{1+4r^2}$, and the bounds are $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. We also have to change the $du\, dv$ into $r\, dr\, d\theta$, and the integral then becomes

$$\begin{aligned}\iint_S 1\, dS &= \iint_R 1\,\sqrt{1+4u^2+4v^2}\, du\, dv \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1+4r^2}\; r\, dr\, d\theta\end{aligned}$$

The inner one is

$$\int_{r=0}^{1} r\sqrt{1+4r^2}\, dr = \frac{1}{12}(1+4r^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}}{12} - \frac{1}{2} = \frac{5\sqrt{5}-1}{12}.$$

The outer is then

$$\int_{\theta=0}^{2\pi} \frac{5\sqrt{5}-1}{12} = \frac{(5\sqrt{5}-1)\pi}{6}.$$

That's the surface area.

Problem 4. *Let S be the top half of a sphere of radius 2 centered at the origin.*

a) *Give a parametrization of S; be sure to include bounds on u and v.*

We're going to use the spherical coordinates θ and ϕ as our coordinates. Let's take u to be ϕ and v to be θ. We're working on the surface S, and so ρ is always going to be 1.

This gives

$$\mathbf{r}(u,v) = \langle 2\sin u\cos v, 2\sin u\sin v, 2\cos u\rangle\,.$$

In this case, we want $0 \le u \le \pi/2$ and $0 \le v \le 2\pi$.

b) *Compute the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v}$.*

We have

$$\begin{aligned}\mathbf{t}_u &= \frac{\partial r}{\partial u} = \langle 2\cos u\cos v, 2\cos u\sin v, -2\sin u\rangle \\ \mathbf{t}_v &= \frac{\partial r}{\partial v} = \langle -2\sin u\sin v, 2\sin u\cos v, 0\rangle \\ \mathbf{t}_u \times \mathbf{t}_v &= 4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\sin u\cos u \\ |\mathbf{t}_u \times \mathbf{t}_v| &= \cdots = 4\sin u.\end{aligned}$$

c) *Compute the surface area of this portion of the sphere. Does your answer match the one you have learned before?*

The surface area is given by

$$\begin{aligned}\iint_S 1\,dS &= \int_{v=0}^{2\pi}\int_{u=0}^{\pi/2} 1\,(4\sin u)\,du\,dv\\ &= \int_{v=0}^{2\pi} 4\,dv = 8\pi.\end{aligned}$$

The formula we know is that the surface area of a sphere is $4\pi R^2$. In this case the radius is 2, so the surface area of a sphere of radius 2 is 16π. Since we only have half a sphere, our answer of 8π makes sense.

d) *Compute the average value of the angle ϕ on this part of the sphere. What is the average latitude of a point in the northern hemisphere?*

In this case, we would want to use $\dfrac{1}{\text{area(S)}}\iint_S \phi\,dS$. The set-up is the same as above, except we need to change the function we're integrating to u (which is how to express ϕ in uv-coordinates).

$$\iint_S 1\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{\pi/2} u\,(4\sin u)\,du\,dv.$$

The inner integral requires an integration by parts, which I'm going to omit. The value is 4. The outer integral is then $\int_{v=0}^{2\pi} 4\,dv = 8\pi$.

We know that the area is also 8π, so the average value of ϕ is just $\phi = 1$. That's one radian, which is about 57.29 degrees. Remember that ϕ is $90°$ minus the latitude, rather than the latitude itself. So the corresponding average latitude is $\approx 90 - 57.29 = 32.70$.

This seems plausible: although there are points with every latitude between 0 and 90 degrees, there are a lot more points at small latitudes than large ones, so the average ought to be less than half of 90. This is roughly the latitude of Dallas.

There's a little global perspective: most points on the earth are closer to the equator than Dallas, and almost the entire United States (except Hawaii and some of the most southerly parts of the south) is further north than the global average.