Math 210 (Lesieutre)
14.6: Surface integrals
April 17, 2017

Problem 1. *Evaluate the indicated flux integrals* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *without making any computations.*

a) *S is the unit sphere, $\mathbf{F}=\mathbf{k}$ is a constant upward field.*

Imagine that this is the flow of fluid in a 3d region. The field is constant, so the amount going into the sphere is equal to the amount going out. The flux is 0.

b) *S is the unit sphere, $\mathbf{F}=\langle x,y,z\rangle$.*

The flow is always directly out of the sphere. Since on the sphere $\mathbf{F}$ is a unit vector, when we dot with the unit normal $\mathbf{n}$ we are going to get 1 for all points. So the integral is $\iint_S 1\,dS$, which is the surface area, which is 4π.

Problem 2. *Compute the flux of the field $\langle 2z,3,2x\rangle$ across the top face of a tetrahedron with vertices at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. (Hint: the equation for the top is $z=1-x-y$.)*

This one is given to us as a graph, so it should be easy to parametrize. We'll use

$$\mathbf{r}(u,v)=\langle u,v,1-u-v\rangle\,.$$

The bounds are a little tricky. We want u and v to vary over the base of this tetrahedron, just like if we were trying to do a triple integral over the region (there was one like this on the last test). So our bounds are going to be

$$\int_{u=0}^{1}\int_{v=0}^{1-u}.$$

We also need the normal vector, which is

$$\begin{aligned}\mathbf{t}_u&=\langle 1,0,-1\rangle\\ \mathbf{t}_v&=\langle 0,1,-1\rangle\\ \mathbf{t}_u\times\mathbf{t}_v&=\langle 1,1,1\rangle\,.\end{aligned}$$

We're now in position to actually compute the integral. Plug in our parametrization to the formula for the field: $\langle 2z,3,2x\rangle=\langle 2-2u-2v,3,2u\rangle$.

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS&=\int_{u=0}^{1}\int_{v=0}^{1-u}\langle 2-2u-2v,3,2u\rangle\cdot\langle 1,1,1\rangle\;dv\,du\\ &=\int_{u=0}^{1}\int_{v=0}^{1-u}5-2v\,dv\,du.\end{aligned}$$

Inner:

$$\int_{v=0}^{1-u} 5 - 2v\, dv = 5v - v^2 \Big|_0^{1-u} = 5(1-u) - (1-u)^2 = 4 - 3u - u^2.$$

Outer:

$$\int_{u=0}^{1} 4 - 3u - u^2 = \frac{13}{6}.$$

Problem 3. *Let V be a cylinder with base at the origin, radius 3, and height 4, and consider the vector field $\mathbf{F} = \langle z^2, 3y, 1\rangle$.*

a) *Compute the flux of $\mathbf{F}$ across S_1, the outer wall of the cylinder.*
Looking at our table, we have $\mathbf{r}(u,v) = \langle 3\cos u, 3\sin u, v\rangle$, with $0 \le u \le 2\pi$ and $0 \le v \le 4$. The normal vector is $\mathbf{t}_u \times \mathbf{t}_v = \langle 3\cos u, 3\sin u, 0\rangle$.
When we plug in our parametrization to the vector field, we get $\mathbf{F} = \langle z^2, 3y, 1\rangle = \langle v^2, 9\sin u, 1\rangle$. That's all we need to know to compute the integral.

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS &= \int_{v=0}^{4}\int_{u=0}^{2\pi} \langle v^2, 9\sin u, 1\rangle \cdot \langle 3\cos u, 3\sin u, 0\rangle\; du\, dv \\
&= \int_{v=0}^{4}\int_{u=0}^{2\pi} 3v^2\cos u + 27\sin^2 u\, du\, dv \\
&= \int_{v=0}^{4} 27\pi\, dv = 108\pi.
\end{aligned}$$

b) *Compute the flux of $\mathbf{F}$ across S_2, the top of the cylinder. (Use an upward-pointing normal vector.)*
Here the surface is just $\mathbf{r}(u,v) = \langle u, v, 4\rangle$, since the top is at height 4. We're going to use the upward normal vector $\langle 0,0,1\rangle$. The region is R, a circle of radius 3.

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_R \langle 16, 4v, 1\rangle \cdot \langle 0,0,1\rangle\; du\, dv \\
&= \iint_R 1\, du\, dv = \text{area}(R) = 9\pi.
\end{aligned}$$

If that had come out to be a hard integral, we'd probably want to switch to polar. Luckily, there was no need.

c) *Compute the flux of $\mathbf{F}$ across S_3, the bottom of the cylinder. (Use a downward-pointing normal vector.)*
This is almost identical to the previous one. The surface is just $\mathbf{r}(u,v) = \langle u, v, 0\rangle$, since the bottom is at height 0. We're going to use the downward normal vector $\langle 0, 0, -1\rangle$. The region

is R, a circle of radius 3.

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \langle 0, 4v, 1 \rangle \cdot \langle 0, 0, -1 \rangle \; du\, dv$$
$$= \iint_R -1\, du\, dv = -\text{area}(R) = -9\pi.$$

d) *What is the total outward flux across the cylinder?*

It's $\oiint_S \mathbf{F} \cdot \mathbf{n}\, dS = 27\pi + 9\pi - 9\pi = 27\pi.$

Why did I make us go through all this? Soon we'll learn the divergence theorem, that says

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

We found that the left side is 27π. On the right, the divergence is $\nabla \cdot \mathbf{F} = 0 + 3 + 0 = 3$, and so

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3\,\text{volume}(V) = 3(36\pi) = 108\pi.$$

So the divergence theorem checks out, at least in this example.