Math 210 (Lesieutre)
14.7: Stokes' theorem, 1
April 21, 2017

Problem 1. *Let S be the portion of the paraboloid $z = 1 - x^2 - y^2$ lying above the plane $z = 0$. Check Stokes' theorem for the field $\mathbf{F} = \langle x, y, z \rangle$.*

Let's do the line integral first. The boundary of S is the unit circle in the xy-plane, and so a parametrization is

$$\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle,$$

with $0 \le t \le 2\pi$. This has

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle.$$

Plugging that in, we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t, \sin t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \; dt = \int_0^{2\pi} 0 \, dt = 0.$$

How about the surface integral? The curl is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (0-0)\mathbf{k} = \langle 0, 0, 0 \rangle.$$

The other side of Stokes' theorem is then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S \mathbf{0} \cdot \mathbf{n} \, dS = \iint_S 0 \, dS = 0.$$

Luckily we don't have to actually parametrize the paraboloid this time, since we're integrating 0. Make sure you know how to! There was one of these a couple worksheets ago.

Problem 2. *Let S be the top half of a sphere of radius 2. Compute $\iint_S \mathbf{k} \cdot \mathbf{n} \, dS$.*

(Hint: for the field $\mathbf{F} = \langle 0, x, 0 \rangle$, we get $\nabla \times \mathbf{F} = \langle 0, 0, 1 \rangle$.)

If there's one thing that the last few days have taught us, it's that computing flux integrals across a sphere is a generally unpleasant undertaking: the parametrization has too many trig functions to keep track of. Luckily for us, $\nabla \times \mathbf{F} = \langle 0, 0, 1 \rangle$, and so Stokes' theorem says that

$$\iint_S \mathbf{k} \cdot \mathbf{n} \, dS = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is a path around the boundary of C. In this case, the boundary is a circle of radius 2 in the xy-plane, which is easy to deal with.

This time things are oriented so that we want to go counterclockwise around a circle of radius 2. A parametrization is

$$\begin{aligned} \mathbf{r}(t) &= \langle 2\cos t, 2\sin t, 0\rangle \\ \mathbf{r}'(t) &= \langle -2\sin t, 2\cos t, 0\rangle \,. \end{aligned}$$

Then

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_{t=0}^{2\pi} \langle 0, 2\cos t, 0\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle \; dt \\ &= \int_{t=0}^{2\pi} 4\cos^2 t\, dt = \int_{t=0}^{2\pi} 4\left(\frac{1+\cos 2t}{2}\right)\; dt \\ &= \int_{t=0}^{2\pi} 2 + 2\cos 2t\, dt = (2t - \sin 2t)\,\Big|_0^{2\pi} = 4\pi. \end{aligned}$$

Problem 3. *What does Stokes' theorem tell us in the case that $\nabla\times\mathbf{F} = 0$? Does this seem plausible?*

If $\nabla\times\mathbf{F} = 0$ then certainly

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = 0.$$

Stokes' theorem tells us that this means

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0,$$

where C is a curve around the boundary of S. But of course we already knew that: $\nabla\times\mathbf{F} = 0$ means that $\mathbf{F}$ is conservative, and because C is a closed loop, the fundamental theorem for line integrals tells us that the integral around a closed loop is 0.

Problem 4. *Let S be a cylinder of height 2 and radius 1 centered at the origin, not including either of the ends. This region has two boundary components. Find an orientation for each, and verify Stokes' theorem for the field $\mathbf{F} = \langle yz, -xz, 0\rangle$.*

Assume the normal vector to the field is pointing outward, we want we to go clockwise around the top edge and counterclockwise around the bottom edge. The top edge is parametrized by

$$\mathbf{r}(t) = \langle \cos t, -\sin t, 1\rangle$$

(note: by "centered at" I mean the cylinder extends up to $z = 1$ and down to $z = -1$). Then

$$\mathbf{r}'(t) = \langle -\sin t, -\cos t, 0\rangle \,.$$

We get

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle (-\sin t)(1), (\cos t)(1), 0 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle \; dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0 \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

The bottom edge is parametrized by

$$\mathbf{r}(t) = \langle \cos t, \sin t, -1 \rangle .$$

Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle .$$

The integral is

$$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle (\sin t)(-1), -(\cos t)(-1), 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \; dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0 \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

So the total circulation is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi + 2\pi = 4\pi.$$

We need to check that this is equal to the other side of Stokes' theorem, which is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

First step is to find the curl of $\mathbf{F}$. For that, I get

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & 0 \end{vmatrix} = (0 - (-x))\mathbf{i} - (0 - y)\mathbf{j} + (-z - z)k = x\mathbf{i} + y\mathbf{j} = \langle x, y, -2z \rangle .$$

We've seen before that for a cylinder of radius a we have a parametrization

$$\mathbf{r}(u, v) = \langle a \cos u, a \sin u, v \rangle$$

which here is just

$$\mathbf{r}(u, v) = \langle \cos u, \sin u, v \rangle$$

since $a = 1$. The bounds are $0 \le u \le 2\pi$ and $-1 \le v \le 1$.

$$\mathbf{n} = \langle a \cos u, a \sin u, 0 \rangle ,$$

which in this case gives

$$\mathbf{n} = \langle \cos u, \sin u, 0 \rangle .$$

Then plugging in x, y, and z to $\nabla \times \mathbf{F}$, we get

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \int_{u=0}^{2\pi} \int_{v=-1}^{1} \langle \cos u, \sin u, 0 \rangle \cdot \langle \cos u, \sin u, 0 \rangle \; dv\, du \\
&= \int_{u=0}^{2\pi} \int_{v=-1}^{1} \cos^2 u + \sin^2 u + 0 \, dv\, du \\
&= \int_{u=0}^{2\pi} \int_{v=-1}^{1} 1 \, dv\, du \\
&= (2\pi)(2) = 4\pi.
\end{aligned}$$

This matches our first answer.