Math 210 (Lesieutre)
14.6/7: Stokes' and divergence theorems
April 21, 2017

Problem 1. *Let S be a cylinder of height 2 and radius 1 centered at the origin, not including either of the ends. This region has two boundary components. Find an orientation for each, and verify Stokes' theorem for the field* $\mathbf{F} = \langle yz, -xz, 0 \rangle$.

Assume the normal vector to the field is pointing outward, we want we to go clockwise around the top edge and counterclockwise around the bottom edge. The top edge is parametrized by

$$\mathbf{r}(t) = \langle \cos t, -\sin t, 1 \rangle$$

(note: by "centered at" I mean the cylinder extends up to $z = 1$ and down to $z = -1$). Then

$$\mathbf{r}'(t) = \langle -\sin t, -\cos t, 0 \rangle .$$

We get

$$\begin{aligned}\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle (-\sin t)(1), (\cos t)(1), 0 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle \; dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0 \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.\end{aligned}$$

The bottom edge is parametrized by

$$\mathbf{r}(t) = \langle \cos t, \sin t, -1 \rangle .$$

Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle .$$

The integral is

$$\begin{aligned}\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle (\sin t)(-1), -(\cos t)(-1), 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \; dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0 \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.\end{aligned}$$

So the total circulation is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi + 2\pi = 4\pi.$$

We need to check that this is equal to the other side of Stokes' theorem, which is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

First step is to find the curl of $\mathbf{F}$. For that, I get

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & 0 \end{vmatrix} = (0 - (-x))\mathbf{i} - (0 - y)\mathbf{j} + (-z + z)k = x\mathbf{i} + y\mathbf{j} = \langle x, y, 0 \rangle .$$

We've seen before that for a cylinder of radius a we have a parametrization

$$\mathbf{r}(u, v) = \langle a\cos u, a\sin u, v \rangle$$

which here is just

$$\mathbf{r}(u, v) = \langle \cos u, \sin u, v \rangle$$

since $a = 1$. The bounds are $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

$$\mathbf{n} = \langle a\cos u, a\sin u, 0 \rangle ,$$

which in this case gives

$$\mathbf{n} = \langle \cos u, \sin u, 0 \rangle .$$

Then plugging in x, y, and z to $\nabla \times \mathbf{F}$, we get

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \int_{u=0}^{2\pi} \int_{v=-1}^{1} \langle \cos u, \sin u, 0 \rangle \cdot \langle \cos u, \sin u, 0 \rangle \; dv\, du \\ &= \int_{u=0}^{2\pi} \int_{v=-1}^{1} \cos^2 u + \sin^2 u + 0\, dv\, du \\ &= \int_{u=0}^{2\pi} \int_{v=-1}^{1} 1\, dv\, du \\ &= (2\pi)(2) = 4\pi. \end{aligned}$$

This matches our first answer.

Problem 2. *Find the flux of the field* $\mathbf{F} = \langle x + y\sin z, xz, 4z \rangle$ *across a sphere of radius* 2 *centered at the origin.*

This would be a pretty bad integral to do directly. Instead, we're going to use the divergence theorem, which lets us convert it into a triple integral over the inside of the sphere.

The divergence theorem tells us that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

So we need to know the divergence, $\nabla \cdot \mathbf{F} = 1 + 0 + 4 = 5$. This reduces our problem to evaluating the triple integral

$$\iiint_D 5\, dV = 5\,\mathrm{volume}(D) = 5(4/3\pi(2)^3) = \frac{160\pi}{3}.$$

Problem 3. *Let S be portion of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane. Compute the flux of the vector field from Problem 2 across S. (Hint: the divergence theorem will make your life easier.)*

The surface S isn't a closed surface, so it's not so clear how to use the divergence theorem here. Let S' be a "cap" on the bottom of the paraboloid, which makes it into a closed surface. Now take D to be the 3d region bounded by the paraboloid and the xy-plane.

The divergence theorem tells us that

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS.$$

The first of these we'll be able to compute without too much trouble. The second of these is what we want to know, and it's a big mess. The third one is also not so bad. So the plan is to compute the first and third and use them to find the second.

Let's do first things first and take the triple integral. We are trying to integrate the function 3. We've done some integrals on paraboloids before; it'll be easiest to use cylindrical coordinates and remember that $z = 1 - x^2 - y^2 = 1 - r^2$.

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{z=0}^{1-r^2} 5\, r\, dr d\theta = \cdots = \frac{5\pi}{2}.$$

Next we want to calculate $\iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS$. We'll use polar: use u is θ and v is r. Then our surface is

$$\begin{aligned} x(u,v) &= v \cos u \\ y(u,v) &= v \sin u \\ z(u,v) &= 0. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{t}_u &= \langle -v \sin u, v \cos u, 0 \rangle \\ \mathbf{t}_v &= \langle \cos u, \sin u, 0 \rangle \\ \mathbf{t}_u \times \mathbf{t}_v &= \langle 0, 0, -v \rangle. \end{aligned}$$

We need to be a little careful here: is this the right normal vector? Since S' is the bottom of the 3d region, and we want an outward normal, we want it to point downward, which is what we got.

Plugging in x, y, and z to $\mathbf{F} = \langle x + y \sin z, xz, 4z \rangle$, we just get $\langle x, 0, 0 \rangle$. That's easy!

So the flux across the bottom is now

$$\iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{v=0}^{1} \int_{u=0}^{2\pi} \langle v \cos u, 0, 0 \rangle \cdot \langle 0, 0, -v \rangle\, du\, dv = \int_{v=0}^{1} \int_{u=0}^{2\pi} 0\, du\, dv = 0.$$

So we've found

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{F}\,dV &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S'} \mathbf{F}\cdot\mathbf{n}\,dS\\ \frac{5\pi}{2} &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS + 0\\ \iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \frac{5\pi}{2}.\end{aligned}$$

It was a pain, but definitely less work than doing the integral directly.

Problem 4. *Check that the divergence theorem is true by computing both sides for the field* $\mathbf{F} = \langle x, 2y, 3z\rangle$ *and the region* $D = \{(x,y,z) : x^2+y^2+z^2 \le 9\}$.

Remember the theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

This is a sphere of radius 3, and so (using the formula from Table 14.3)

$$\begin{aligned}\mathbf{r}(u,v) &= \langle 3\sin u\cos v, 3\sin u\sin v, 3\cos u\rangle\\ \mathbf{t}_u\times\mathbf{t}_u &= \left\langle 9\sin^2 u\cos v, 9\sin^2 u\sin v, 9\sin u\cos u\right\rangle\\ \iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_{u=0}^{\pi}\int_{v=0}^{2\pi} \langle 3\sin u\cos v, 6\sin u\sin v, 9\cos u\rangle\cdot\\ &\qquad \left\langle 9\sin^2 u\cos v, 9\sin^2 u\sin v, 9\sin u\cos u\right\rangle\,dv\,du\\ &= \int_{u=0}^{\pi}\int_{v=0}^{2\pi} 27\cos^2(v)\sin^3(u) + 54\sin^3(u)\sin^2(v) + 81\cos^2(u)\sin(u)\,dv\,du\\ &= \cdots = 36\pi + 72\pi + 108\pi = 216\pi.\end{aligned}$$

(I'm leaving on the work on the integrals, but they're actually pretty harmless.)

On the other hand, the divergence of the field is $1+2+3=6$, and so

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D 6\,dV = 3(4/3\pi(3)^3) = 216\pi.$$

So the divergence theorem checks out.