Math 210 (Lesieutre)
Exam review #1
April 28, 2017

Problem 1. *Try to recall the main integration theorems from this chapter.*

a) *Fundamental theorem for line integrals.*

This one says that if $\mathbf{F}$ is a conservative field, so that $\mathbf{F} = \nabla\phi$ for some scalar function ϕ, and if C is a curve, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A),$$

where B is the endpoint of C and A is the beginning point of C.

b) *Green's theorem, circulation form.*

This time we have a region R and a curve C going around R, counterclockwise. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA.$$

Here $\operatorname{curl} \mathbf{F}$ is the curl (duh). If we have a 2D field $\mathbf{F} = \langle f(x, y), g(x, y) \rangle$, then $\operatorname{div} \mathbf{F} = g_x - f_y$.

c) *Green's theorem, flux form.*

This is similar to the previous one. It says that

$$\oint_C \mathbf{F} \cdot \mathbf{n} = ds = \iint_R \operatorname{div} \, dA.$$

Here $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$ is the divergence. If we have a 2D field $\mathbf{F} = \langle f(x, y), g(x, y) \rangle$, then $\operatorname{div} \mathbf{F} = f_x + g_y$.

d) *Stokes' theorem*

This is a 3D thing. We have a surface S in 3D, and the boundary of the surface is a curve C (oriented appropriately – see the previous sheet).

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

e) *Divergence theorem*

The divergence theorem related a triple integral to a double integral. Let D be a 3D region, with boundary a surface S. Then

$$\iiint_D (\nabla \cdot \mathbf{F}) \, dV = \oiint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

Problem 2. *Find the equation for the tangent line to the curve* $\mathbf{r}(t) = \langle t, \sin t, 3 \rangle$ *at the point* $(\pi, 0, 3)$.

Any find-a-line question, the strategy is exactly the same. We need to know a point that the line goes through, and the direction vector for the line. In this case, the point is given to us: it's $(\pi, 0, 3)$. So we need the direction. To find the direction of the tangent, we need to take the derivative of $\mathbf{r}(t)$ (remember that that will give the tangent vector to a curve). Since our point is $\mathbf{r}(\pi)$, the tangent vector will be the vector $\mathbf{r}'(\pi)$. I get $\mathbf{r}'(t) = \langle 1, \cos t, 0 \rangle$, and so $\mathbf{r}'(\pi) = \langle 1, -1, 0 \rangle$. So our line has equation

$$\ell(t) = \mathbf{r}_0 + \mathbf{v}t = \langle \pi, 0, 3 \rangle + t \langle 1, -1, 0 \rangle = \langle \pi + t, -t, 3 \rangle .$$

Problem 3. *Let $f(x, y) = x^2 - x + y^2$. Find the absolute maximum and minimum of $f(x, y)$ on a disk of radius 3 centered at the origin.*

This is a classic max/min with a boundary problem. Two parts: find the critical points on the interior, and find the max/min on the boundary.

First, let's do the critical points. We have $f_x = 2x - 1$ and $f_y = 2y$. Solve $f_x = 0$ and $f_y = 0$, we find that the only possibility is $(x, y) = (1/2, 0)$. So that's our first candidate as a global max/min.

Now we need to find the max and min on the boundary. There are a couple ways to handle this. First is the way we usually did it before: parametrize the boundary in terms of t, and find the values of t that are going to maximize or minimize. THe second way is to use Lagrangle multipliers, since this is really a constrained optimization problem: we are trying to find the max/min of $f(x, y) = x^2 - x + y^2$ subject to the constraint $x^2 + y^2 - 9 = 0$.

First, the parametrization method: the curve is given by $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$, where $0 \leq t \leq 2\pi$. Plugging that in to f, we get

$$g(t) = f(\mathbf{r}(t)) = 9 \cos^2 t - 3 \cos t + 9 \sin^2 t = 9 - 3 \cos t.$$

What are the max and min of this, with respect to t? Well, take the derivative:

$$g'(t) = 3 \sin t,$$

which is 0 when $t = 0$ or π. These are possible max/min on the boundary, and they correspond to the points $\mathbf{r}(0) = \langle 3, 0 \rangle$ and $\mathbf{r}(\pi) = \langle -3, 0 \rangle$. The possible interesting points are therefore:

(x, y)	$f(x, y)$
$(1/2, 0)$	$-1/4$
$(3, 0)$	6
$(-3, 0)$	12

This shows that the max is 12, achieved at $(-3, 0)$, and the min is $-1/4$, achieved at $(1/2, 0)$.

We could also have used Lagrange to find the extreme points on the boundary. We want to maximize $f(x, y) = x^2 - x + y^2$ subject to the constraint $g(x, y)x^2 + y^2 - 9 = 0$.

$$\begin{aligned} \nabla f &= \langle 2x - 1, 2y \rangle \\ \nabla g &= \langle 2x, 2y \rangle . \end{aligned}$$

The equation $\nabla f = \lambda \nabla g$ gives us the two equations $2x - 1 = \lambda(2x)$ and $2y = \lambda(2y)$. The third equation is $x^2 + y^2 - 9 = 0$. The second equation is the most promising. It says that $2y(\lambda - 1) = 0$, so either $y = 0$ or $\lambda = 1$. If $y = 0$, the third equation gives $x = 3$ or $x = -3$, giving two points $(x, y) = (3, 0)$ and $(x, y) = (-3, 0)$. In the other case, that $\lambda = 1$, we get $2x - 1 = 2x$, which has no solutions. So we found the only two solutions. These are the same poits we got with the paametrization method, so you'd do the remained of the problem the same way.

Problem 4. *Consider the surface S defined by $z = 1 + x + 2y$ and above the rectangle $[1, 2] \times [2, 3]$.*

a) *Set up an integral to compute the volume below S and above the xy-plane.*

We'd just use

$$\int_{x=1}^{2} \int_{y=2}^{3} 1 + x + 2y \, dy \, dx = \cdots = \frac{15}{2}.$$

b) *Set up an integral for the surface area of S.*

This is more work. We want to compute $\iint_S 1 \, dS$, which mean we have to go through all the rigamarole of parametrization and so on.

The parametrization is $x(u, v) = u$, $y(u, v) = v$, and $z(u, v) = 1 + u + 2v$, with $1 \leq u \leq 2$ and $2 \leq v \leq 3$. This gives

$$\begin{aligned} \mathbf{r}(u, v) &= \langle u, v, 1 + u + 2v \rangle \\ \mathbf{t}_u &= \langle 1, 0, 1 \rangle \\ \mathbf{t}_v &= \langle 0, 1, 2 \rangle \\ \mathbf{t}_u \times \mathbf{t}_v &= \langle -1, -2, 1 \rangle \\ |\mathbf{t}_u \times \mathbf{t}_v| &= \sqrt{6}. \end{aligned}$$

The surface area becomes

$$\begin{aligned} SA = \iint_S 1 \, dS &= \int_{u=1}^{2} \int_{v=2}^{3} 1 \, |\mathbf{t}_u \times \mathbf{t}_v| \, dv \, du \\ &= \int_{u=1}^{2} \int_{v=2}^{3} \sqrt{6} \, dv \, du = \sqrt{6}. \end{aligned}$$

c) *Set up an integral for the flux of* $\langle x^2, y-z, 3\rangle$ *across* S.

We're going to use the stuff that we computed in the previous part of the problem. To compute flux, we substitute in the parametrization to our equation for the vector field:

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_{u=1}^{2}\int_{v=2}^{3} \left\langle u^2, v-(1+u+2v), 3\right\rangle \cdot \langle -1,-2,1\rangle\; dv\,du \\ &= \int_{u=1}^{2}\int_{v=2}^{3} -u^2+2u+2v+5\,dv\,du = \frac{32}{3}.\end{aligned}$$

Problem 5. a) *Consider the integral* $\int_{x=0}^{3}\int_{y=0}^{x^3} xy\,dy\,dx$. *Sketch the region of integration, and reverse the order of the integrals.*

It's the region underneath the graph of $y=x^3$ and above the x-axis, for x between 0 and 3. The top right corner of the region is the point $(3,27)$, and so on this region we're going to have y from 0 to 27. For a given y, the lower bound on x is the graph, and the upper bound is $x=0$. The graph is $y=x^3$, which means $x=\sqrt[3]{y}$. So our integral is going to be:

$$\int_{y=0}^{27}\int_{x=\sqrt[3]{y}}^{3} xy\,dx\,dy.$$

b) *Consider the integral* $\int_{x=0}^{3}\int_{y=0}^{\sqrt{9-x^2}} e^{-x^2-y^2}\,dy\,dx$.. *Convert this integral into polar coordinates.*

There are three things we need to do: convert the bounds to polar, convert the function to polar, and convert $dy\,dx$ to polar. First the bounds. This region is the first quadrant part of a circle of radius 3, so we're going to have $0\le r\le 3$ and $0\le\theta\le\pi/2$. The function is $e^{-x^2-y^2}=e^{-r^2}$. And as always, we're going to use $r\,dr\,d\theta$. So our answer is

$$\int_{x=0}^{3}\int_{y=0}^{\sqrt{9-x^2}} e^{-r^2}\,r\,dr\,d\theta.$$