SOME PROBLEMS YOU SHOULD BE ABLE TO DO

I've attempted to make a list of the main calculations you should be ready for on the exam, and included a handful of the more important formulas. There are no examples here: for that, check my lecture worksheets accompanying the corresponding sections. Please let me know if you notice any mistakes or omissions!

I don't recommend studying exclusively from this list – it is inevitable that I have left something out. Be sure to look at old exams, problem sets, etc. as well.

A few topics we covered only briefly are in italics. It couldn't hurt to look, but don't worry about these too much.

Chapter 11

11.1 and 11.2: Vectors in the plane and vectors in three dimensions.

- ☐ Compute basic vector operations: addition/subtraction, scalar multiplication, magnitude.
- ☐ Know how to interpret these both algebraically and geometrically.
- ☐ Find vector from point P to point Q.
- ☐ Find a unit vector in the direction of a given vector (or a vector with some other specified length)
- ☐ Midpoint of a line segment.
- ☐ Equation of a sphere.
- ☐ *Statics problems (balancing forces).*

11.3: Dot products.

- ☐ Compute the dot product of two vectors algebraically ($a_1b_1 + a_2b_2 + a_3b_3$) and geometrically ($|\mathbf{a}|\,|\mathbf{b}|\cos\theta$)
- ☐ Compute the angle between two vectors
- ☐ Compute and interpret $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ and $\operatorname{scal}_{\mathbf{v}}(\mathbf{u})$:

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}, \quad \operatorname{scal}_{\mathbf{v}} \mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

- ☐ Work done by a constant force.
- ☐ Parallel and normal components of a force.

11.4: Cross products.

- ☐ Compute the cross product of two vectors.
- ☐ Find the area of a triangle with given vertices.
- ☐ Find the area of a parallelogram with given vertices.

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11.5: Lines and curves in space.

- ☐ Parametrize a straight line
- ☐ You always need to know two things: a point $\mathbf{r}_0$ that the line goes through, and a vector $\mathbf{v}$ in the direction of the line (often you will need to do some work to find $\mathbf{v}$, depending on what information is given to you!). Then use the formula $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ (and think about the bounds)
  - ☐ between two given points
  - ☐ through a point and perpendicular to a given plane
  - ☐ through a point and perpendicular to two given lines
  - ☐ tangent to a curve $\mathbf{r}(t)$ at $t = a$
  - ☐ given as the intersection of two planes
- ☐ Parametrize other simple curves (circles)
- ☐ Check whether lines intersect
- ☐ Take a limit (by taking the limit of each component)

11.6: Calculus of vector-valued functions.

- ☐ Find the derivative $\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
- ☐ Find tangent vector to a curve at time t
- ☐ Find unit tangent vector to a curve
- ☐ Integrate a vector-valued function (by integrating each component)

11.7: Motion in space.

- ☐ Find velocity, acceleration, and speed.
- ☐ Find position from velocity and velocity from acceleration, given an initial condition $\mathbf{v}(0)$ or $\mathbf{a}(0)$.
- ☐ *Movement in a gravitational field.*

11.7: Length of curves.

- ☐ The arc length of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

  This is the distance traveled by a particle moving along the curve from $t = a$ to $t = b$.
- ☐ Find arc length in polar.
- ☐ Check whether a path is parametrized by arc length.

Chapter 12

12.1: Planes and surfaces.

- ☐ Find the equation for a plane with normal vector $\langle a, b, c \rangle$ passing through (x_0, y_0, z_0).
- ☐ Find the equation for a plane through three given points
- ☐ Find the equation for a line given as the intersection of two planes.
- ☐ Check whether two planes are orthogonal.
- ☐ *Equations for cylinders*

12.2: Quadric surfaces.

- ☐ Sketch the graph of a quadric surface by drawing the xy-, xz-, and yz-traces, or some other traces parallel to the coordinate planes.
- ☐ Find the intersection of a line with a quadric surface.

12.3: Limits of functions of several variables.

- ☐ Compute the limit of a function of two variables.
- ☐ Use the two-path test to show that a limit does not exist.

12.4 & 12.5: Partial derivatives.

- ☐ Compute the partial derivative of a function $f(x, y)$
- ☐ Compute the four second-order partial derivatives of a function
- ☐ Use the chain rule to compute partial derivatives of functions of two or three variables.
- ☐ Apply implicit differentiation to an expression $F(x, y) = 0$

12.6: Directional derivatives and the gradient.

- ☐ Compute (and interpret) the directional derivative $D_{\mathbf{u}}f$
- ☐ Find the gradient ∇f.
- ☐ Find direction of fastest ascent/descent and the rate of fastest ascent/descent for a function $f(x, y)$.
- ☐ Find directions of 0 increase.
- ☐ Sketch level curves of a function, and tangent directions to level curves.

12.7: tangent planes and linear approximation.

- ☐ Find tangent plane to implicit surface $F(x, y, z) = 0$ at a point (x_0, y_0, z_0).
- ☐ Find tangent plane to explicit surface $z = f(x, y)$ at a point (x_0, y_0, z_0).
- ☐ Find the linear approximation to $f(x, y)$ near a point (a, b) and use this to approximate values of f.
- ☐ *Work with differentials.*

12.8: Maxima and minima.

- ☐ Find the critical points of a function.
- ☐ Use the second derivative test to classify the critical points as max/min/saddle.
- ☐ Find the global max/min of a function $f(x, y)$ on a region R:
  1. Find critical points inside R
  2. Find relative max/min on the boundary
  3. Make a list of all the "interesting points" and compute values of f there to find the true max and min.

12.9: Lagrange multipliers.

- ☐ Use Lagrange multipliers to find the maxima and minima of $f(x, y)$ subject to the constraint $g(x, y) = 0$
- ☐ Key formula: $\nabla f(x, y) = \lambda \nabla g(x, y)$.
  Translate this into three equations in the three variables x, y, and λ and solve
- ☐ Translate a word problem into a constrained optimization problem.

Chapter 13

13.1: Double integrals.

- ☐ Set up and compute the double integral of a function $f(x, y)$ on a rectangular region $a \leq x \leq b$, $c \leq y \leq d$.
- ☐ Find the average value of $f(x, y)$ on such a region.
- ☐ Fubini's theorem: you can change the order of the integral.

13.2: Double integrals over other regions.

- ☐ Evaluate a double integral where the inner bounds depend on the outer variable.
- ☐ Sketch the region of integration for a double integral, given the bounds.
- ☐ Give the bounds on a double integral, given a description of the region.
- ☐ Change the order of integration in a double integral over a non-rectangular region.

13.3: Double integrals in polar coordinates.

- ☐ Sketch the region of integration for a double integral in polar
- ☐ Evaluate a double integral in polar.
- ☐ Convert a double integral in rectangular coordinates into polar:
  1. Write the bounds in polar
  2. Write the function in polar (substitute $r \cos\theta$ for x, $r \sin\theta$ for Y
  3. Write $r\, dr\, d\theta$ instead of $dx\, dy$.

13.4: Triple integrals.

- ☐ Evaluate a triple integral.
- ☐ Set up the bounds for a triple integral in rectangular coordinates (important shapes: rectangular regions, tetrahedral regions)
- ☐ Change the order of integration in a triple integral.

13.5: Cylindrical spherical coordinates.

- ☐ Find the rectangular coordinates for a point given (r, θ, z)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

- ☐ Find the cylindrical coordinates for a point given (x, y, z):
- ☐ Set up the bounds for a triple integral in cylindrical coordinates (Important shapes: cylinder, paraboloid.)
- ☐ Convert a rectangular integral into cylindrical coordinates and evaluate (usual three steps: convert the bounds, convert the function, use $r\, dr\, d\theta\, dz$).
- ☐ Find the rectangular coordinates for a point given (ρ, θ, ϕ):

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

- ☐ Set up the bounds for a triple integral into spherical coordinates (Important shapes: spheres, hemispheres, ice cream cones.)
- ☐ Convert a rectangular integral into spherical coordinates and evaluate (usual three steps: convert the bounds, convert the function, use $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

13.7: Change of variables in multiple integrals.

☐ Given a change of coordinates $x = g(u, v)$, $y = h(u, v)$, compute the Jacobian $J(u, v)$:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(x,y)}{\partial(u,v)}.$$

☐ Change an integral in terms of x and y to an integral in terms of u and v:

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v))\,|J(u,v)|\;dA.$$

☐ Find the uv bounds for an integral from the xy bounds (often helpful; convert the equation for each edge of the region into an equation involving uv, and sketch the corresponding region in the uv-plane).

Chapter 14

14.1: Vector fields.

☐ Sketch a vector field from the equation.
☐ Write a formula for a vector field given a description.
☐ Compute the gradient field $\mathbf{F} = \nabla f(x, y)$ for a function $f(x, y)$

14.2: Line integrals.

☐ Compute the line integral $\int_C f(x, y)\,ds$ of a scalar function along a path.

- Method: parametrize the path C by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Use bounds as parametrization as bounds on integral, plug in x and y to f, and use $|\mathbf{r}'(T)|\;dt$ for the ds:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\;|\mathbf{r}'(t)|\;dt.$$

☐ Compute the line integral of a $\int_C \mathbf{F}\cdot d\mathbf{r}$ of a vector function along a path (i.e. a circulation integral).

- Parametrize C by $\mathbf{r}(t)$, then plug in x and y to $\mathbf{F}$, and dot that with $\mathbf{r}'(t)$ to get a function of t.

☐ Compute the flux of a vector field across a path C

- Same method, but use $\mathbf{n} = \langle y(t), -x(t)\rangle$ instead of $\mathbf{r}'(t)$ in the above.

14.3: Conservative vector fields.

☐ Check whether a 2D vector field $\mathbf{F} = \langle f, g\rangle$ is conservative.
☐ Check whether a 3D vector field $\mathbf{F} = \langle f, g, h\rangle$ is conservative.
☐ Find a potential function $\phi(x, y)$ for a conservative vector field.
☐ Use the fundamental theorem for line integrals to compute the line integral of a conservative field along a path C:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(\text{end}) - \phi(\text{begin}).$$

Note: integral is indepedent of path.

14.4: Green's theorem.

- ☐ Compute the divergence and curl of a 2D vector field (both are scalar functions, not vetor fields).
- ☐ Be able to apply both versions of Green's theorem to double integrals over a region R:
  - Circulation form:
  $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F}\, dA$$
  - Flux form:
  $$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$
- ☐ Use Green's theorem to turn a line integral over a complicated path from A to B into a line integral over a simpler path from A to B and a double integral over the region between the two paths.

14.5: Divergence and curl.

- ☐ Compute the divergence of a 3D vector field $\nabla \cdot \mathbf{F}$ (this is a scalar function).
- ☐ Compute the curl of a 3D vector field:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$
$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

  NB: This is another vector field.

14.6: Surface integrals.

- ☐ Parametrize a surface in 3D
  1. Cylinder of radius a; $x = a\cos u$, $y = a\sin u$, $z = v$
  2. Sphere of radius a: $x = a\sin u\cos v$, $y = a\sin u\sin v$, $z = a\cos u$.
  3. Graph $z = f(x, y)$ (for example, $z = 2(x^2 + y^2)$): $x = u$, $y = v$, $z = f(u, v)$.
- ☐ Next, know how to compute the tangent vectors $\mathbf{t}_u$, $\mathbf{t}_v$, and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$.
- ☐ Compute surface integrals of scalar functions: $\iint_S f(x, y, z)\, dS$. Steps to convert to a double integral in u, v:
  1. The bounds are the bounds for your parametrization
  2. Rewrite the function in terms of u and v (plug in your parametrization for x, y, and z)
  3. Use $|\mathbf{t}_u \times \mathbf{t}_v|\, du\, dv$ for dS.
- ☐ Compute flux integrals of vector fields across a surface: $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. Steps to convert to a double integral in u, v:
  1. The bounds are the bounds for your parametrization
  2. Rewrite the function in terms of u and v (plug in your parametrization for x, y, and z)
  3. Plug in x, y, z from parametrization to $\mathbf{F}$
  4. Dot that with $\mathbf{t}_u \times \mathbf{t}_v$ and integrate the result $du\, dv$.

14.7: Stokes' theorem.

- ☐ Be able to apply Stokes' theorem, either to turn a computation of the flux of a curl into a circulation integral, or to turn a circulation integral into the flux of a curl.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

- ☐ Know how to compute both sides!
- ☐ Know which way to go around the curve C to make this work (right-hand rule).
- ☐ Apply Stokes' theorem on regions with multiple boundaries.

14.8: Divergence theorem.

- ☐ Be able to apply divergence theorem, in either direction.

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

- ☐ Know how to compute both sides!