Announcements

- Quizzes not all graded yet. Will return F - sorry!

- Midterm in two weeks: stay tuned for details.
  Let me know ASAP if you need a make-up.

- Today: finding eigenvalues / eigenvectors.

How to find eigenvalues/eigenvectors?

A an $n \times n$ matrix

Remember: eigenvector $\vec{x}$ is vector so $A\vec{x} = \lambda\vec{x}$

λ called eigenvalue.

Last time: given A, given λ, find all eigenvectors with eigenvalue λ by finding $\text{Nul}(A - \lambda I)$.

if $\vec{x}$ in nullspace, then $(A - \lambda I)\vec{x} = \vec{0}$

$$A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$A\vec{x} = \lambda\vec{x} \Rightarrow \vec{x} \text{ eigenvector.}$$

But! for most λ, there are no eigenvectors (other than $\vec{0}$)

how to find the λ's that work?

want: find λ so $A - \lambda I$ has nonzero nullspace.

$\rightarrow$ so find λ that make $\det(A - \lambda I) = 0$.

just compute $\det(A - \lambda I)$, in terms of λ.

$\det(A - \lambda I)$ is polynomial in terms of λ, called "characteristic polynomial"

roots of this polynomial are the eigenvalues.

ex

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

characteristic poly is

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$= (1-\lambda)(4-\lambda) - (2)(2) = 4 - 5\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 5\lambda = \lambda(\lambda - 5).$$

roots are $\lambda = 0$
$\lambda = 5$ ← these are eigenvalues.

now we can find corresponding eigenvectors, as on Monday.

do each eigenval separately.

$\lambda = 0$: $A - \lambda I = A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

want $(A - \lambda I)\vec{x} = 0$.

$\rightsquigarrow \vec{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. check:

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ +1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \vec{x}$ ✓

$\lambda = 5$. $A - \lambda I = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$.

$(A - \lambda I)\vec{x} = \vec{0} \rightsquigarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{x}$.

check: $A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

~~$(A-\lambda I)\vec{x}$~~ $= 5\vec{x}$ ✓

If A is triangular, life is easy:

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

char. poly:

$$\det(A - \lambda I) = \det\left[\begin{pmatrix} 1 & 3 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} 1-\lambda & 3 & 7 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

determinant is $(1-\lambda)(2-\lambda)(3-\lambda)$ (multiply diagonal entries to get det)

char poly.

roots are $\lambda = 1, 2, 3$.

for any triangular matrix, the eigenvalues are just the numbers on the diagonal. then find eigenvectors as before.

For a non-triangular matrix, eigenvalues will have square roots, fractions, and worse!

Important fact:

if A is an $n \times n$ matrix,

and $\vec{v}_1, \dots, \vec{v}_r$ are eigenvectors, ~~the~~ with different eigenvalues,

then $\vec{v}_1, \dots, \vec{v}_r$ are linearly independent.

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most $n \times n$ matrices have n different eigenvalues.

if you take one $\vec{v}_i$ for each λ_i, you get a basis: "eigenbasis"

(Didn't do a 3×3 one in class; including here for reference)

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & -6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & -6-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (7-\lambda) \det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 6-\lambda \end{pmatrix}$$

$$= (7-\lambda)\big((2-\lambda)(6-\lambda) - 9\big)$$

$$= (7-\lambda)(-21 + 4\lambda + \lambda^2)$$

$$= (7-\lambda)(\lambda-3)(\lambda+7)$$

the eigenvalues are $\lambda = -7, 3, 7$.

for each one, find an eigenvector by solving

$$(A - \lambda I)\vec{x} = \vec{0}.$$