Quick recap

Three ways to write an system of linear equations.

①
$$\begin{aligned} x_1 + 3x_3 &= 7 \\ x_1 + x_2 + x_3 &= 4 \\ 3x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$
"system of linear equations"

②
$$x_1\begin{pmatrix}1\\1\\3\end{pmatrix} + x_2\begin{pmatrix}0\\1\\-2\end{pmatrix} + x_3\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}7\\4\\6\end{pmatrix}$$
"vector equation"

Is $\begin{pmatrix}7\\4\\6\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}$?

③
$$\begin{pmatrix}1&0&3\\1&1&1\\3&-2&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}7\\4\\0\end{pmatrix}$$
"matrix equation"

$$A \quad \vec{x} = \vec{b}$$

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same method to solve all 3:

augmented matrix

$$\underbrace{\left(\begin{array}{ccc|c}1&0&3&7\\1&1&1&4\\3&-2&1&0\end{array}\right)}_{A \quad \vec{b}} \xrightarrow[\text{reduction}]{\text{row}} \left(\begin{array}{ccc|c}1&0&0&1/4\\0&1&0&3/2\\0&0&1&9/4\end{array}\right)$$

1.5: Solution sets of systems of linear equations.

A *homogeneous* system of linear eqns is one of the form:

$$A\vec{x} = \vec{0} \qquad (\vec{b} \text{ is } 0).$$

the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

example: $\underbrace{\begin{pmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{b} = \vec{0}}$

$$3x_1 - 9x_2 + 6x_3 = \underline{0}$$
$$-x_1 + 3x_2 - 2x_3 = \underline{0}$$

for a homogeneous system, these are all 0's

A homogeneous linear system is *always* consistent.

silly reason: $\vec{x} = \vec{0}$ is a solution.

(but sometimes there are others too!)

$$\left(\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right) \xrightarrow{\text{row } 1 \div 3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right) \xrightarrow[\text{1 to row 2}]{\text{add row}} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

basic: x_1

free: x_2, x_3.

general sol: $\begin{cases} x_1 = 3x_2 - 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

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plug in s for x_2, t for x_3.

our solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3s-2t \\ s \\ t \end{pmatrix}$

rewrite this: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

(another way to write down the general solution)

"parametric vector form" for the solution.

What about non-homogeneous systems?

can still write in parametric vector form!

__example__

$$\left(\begin{array}{ccc|c} 3 & -9 & 6 & 6 \\ -1 & 3 & -2 & -2 \end{array}\right) \xrightarrow{\text{row } 1 \times \frac{1}{3}} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ -1 & 3 & -2 & -2 \end{array}\right) \xrightarrow[\text{to row 2}]{\text{add row 1}} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

general solution $\begin{cases} x_1 = 2 + 3x_2 - 2x_3 \\ x_2 = s \\ x_3 = t \end{cases}$.

in parametric vector form,

$$\text{solution is } \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \underbrace{s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{same as homogeneous sol.}}$$

This always happens!

every solution to $A\vec{x} = \vec{b}$ is of the form

$$\vec{x} = \vec{v}_p + \vec{w} \qquad \vec{w} = s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

$\vec{v}_p$: "particular sol" e.g. $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$\vec{w}$: solution to homogeneous $A\vec{x} = \vec{0}$.