Math 310, Lesieutre
Quiz #10
November 18, 2015

Name: ______________________________

1. *Consider the matrix*
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$
Find the (complex) eigenvalues and eigenvectors.

This matrix is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so you can get the answer just by remembering the answer for that kind of matrix, which is in the lecture notes or the book. Let's do it the long way, just for the sake of having another example worked out.

We have
$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) + 1.$$

We want $(\lambda - 1)^2 + 1 = 0$, so $(\lambda - 1)^2 = -1$ and $\lambda = 1 \pm i$. (You could do this with the quadratic formula if you'd rather.) Let's find the eigenvector for $\lambda = 1 - i$. We get
$$A - \lambda I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}.$$

By the "switch the entries and add a $-$" trick for eigenvectors of 2×2 matrices, we get
$$\mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

To get the eigenvector for the other eigenvalue (which is the conjugate of this eigenvalue), we just take the complex conjugate of the eigenvector, and obtain
$$\mathbf{w} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

2. *Let A be the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. The eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.*
Now consider the differential equation $\mathbf{x}' = A\mathbf{x}$. What is the general solution (in terms of parameters c_1 and c_2?) What is the solution satisfying $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$?

The general solution is just
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}.$$

When we plug in $t = 0$, this is just

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

which we want to be equal to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. The solution is given by $c_1 = 2$ and $c_1 = 0$, so that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t.$$

In other words, $x_1(t) = 2e^t$, $x_2(t) = 2e^t$.

3. *Find the distance between the two vectors*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

We have

$$\mathbf{v} - \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}.$$

The distance between the two is the length of this, which is

$$\sqrt{0^2 + 3^2 + 2^2} = \sqrt{13}.$$