Math 310, Lesieutre
Quiz #11
November 25, 2015

Name: ______________________________

1. *Let* $\mathbf{v}$ *be the vector* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *Let* W *be the subspace of* $\mathbb{R}^3$ *given by all vectors perpendicular to* $\mathbf{v}$.

   (a) *Find a basis for* W *(hint:* W *is the nullspace of the* 1×3 *matrix* $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$*).*
   To find the nullspace of $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, just use parametric vector form as usual. Reduced echelon form of the augmented matrix (with an extra column of 0s) is given by

   $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right]$$

   The variables x_2 and x_3 are free, while x_1 is a pivot. The general solution is

   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   A basis for W is given by the two vectors

   $$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   (b) *Find an orthonormal basis for* W.
   The two vectors we got are not an orthonormal basis: they aren't even orthogonal to each other. So we need to use the Gram–Schmidt process. First we construct an orthogonal basis:

   $$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

   $$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

   This has $\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$, like we want. But it's still not an orthogonal basis: for that, we need to divide each of the vectors by its length.

   $$\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2/3} \end{bmatrix}.$$

(c) *The vector* $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ *is perpendicular to* $\mathbf{v}$*, so it's in the subspace* W*. Write this vector as a linear combinations of your orthonormal basis vectors from (b).*

There's a quick rule for this: $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ where

$$c_1 = \mathbf{v} \cdot \mathbf{u}_1 = -2/\sqrt{2} - 1/\sqrt{2} = -\frac{3\sqrt{2}}{2}$$
$$c_2 = \mathbf{v} \cdot \mathbf{u}_2 = -2/\sqrt{6} + 1/\sqrt{6} - \sqrt{2/3} = -\sqrt{3/2}.$$

So

$$\mathbf{v} = -\frac{3\sqrt{2}}{2}\mathbf{u}_1 - \sqrt{\frac{3}{2}}\mathbf{u}_2.$$

(d) *Let* $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$*. What point on* W *is closest to* $\mathbf{p}$*?*

We want $\text{proj}_W\, \mathbf{p}$, which is given by

$$\text{proj}_W\, \mathbf{p} = (\mathbf{p} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{p} \cdot \mathbf{u}_2)\mathbf{u}_2 = \cdots = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

(e) *Let* A *be the* 3×2 *matrix whose columns are the vectors in your answer to (a). Find a QR decomposition for* A*.*

We have Q given by the columns $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2/3} \end{bmatrix}.$$

Then

$$R = Q^T A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2/3} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{bmatrix}.$$