Math 310, Lesieutre  Name: ______________________________
Quiz #8
October 28, 2015

1. *Consider the two bases $\mathcal{B}$ and $\mathcal{C}$ for $\mathbb{R}^2$ given by the vectors*

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{c}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \ \mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

   (a) *Find the change of basis matrix* $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$.

   The easy way is to just use the formula

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{C}}^{-1}\mathcal{P}_{\mathcal{B}}.$$

   In this case,

$$\mathcal{P}_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$\mathcal{P}_{\mathcal{C}} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$
$$\mathcal{P}_{\mathcal{C}}^{-1} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{(3)(1) - (2)(2)} \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$
$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{C}}^{-1}\mathcal{P}_{\mathcal{B}} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}.$$

   (b) *Suppose that $\mathbf{x}$ is a vector with $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Compute $[\mathbf{x}]_{\mathcal{C}}$.*

   The point of the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ is that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_{\mathcal{B}}.$$

   For the vectors in question, this is just

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

2. *Suppose that $A$ is a $3 \times 5$ matrix. What is the smallest possible dimension of* $\operatorname{Nul} A$*?*

   The column space of $A$ is a subspace of $\mathbb{R}^3$, so its dimension is at most 3. Since the rank theorem tells us that

$$\operatorname{rank} A + \dim \operatorname{Nul} A = 5,$$

   this means that $\dim \operatorname{Nul} A$ is at least 2. You could also do this by thinking a little about what $\operatorname{rref}(A)$ could look like: you can only have one pivot per row, and there are only three rows, so the number of pivots is at most 3. That means there are at least two free variables, which means that $\dim \operatorname{Nul} A$ is at least 2.

3. *A car rental company operates two stations, one in City A and one in City B. Every day, 80% of the cars in A remain in A while the rest move to B, and 90% of the cars in B remain in B, while the rest move to A.*

(a) *Write down the stochastic matrix $M$ describing the movement of cars between the two cities.*

We're told that

| | | From $A$ | From $B$ |
|---|---|---|---|
| To | $A$ | 0.8 | 0.1 |
| | $B$ | 0.2 | 0.9 |

So the matrix $M$ is given by

$$M = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}.$$

(b) *Find the steady state of $M$. Describe in words the meaning of your answer.*

We want $(M - I)\mathbf{x} = \mathbf{0}$. Notice that

$$M - I = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}.$$

A solution is given by $\mathbf{x} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$. We want our $\mathbf{x}$ to be a state vector (the entries sum to 1), so we divide everything by 0.3 and get

$$\mathbf{x} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$