

1. Let A be the matrix

$$A = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}.$$

(a) Find the characteristic polynomial and eigenvalues of A .

We have

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -3 - \lambda & 1 \\ -6 & 2 - \lambda \end{bmatrix} = (-3 - \lambda)(2 - \lambda) - (-6) \\ &= -6 + \lambda + \lambda^2 + 6 = \lambda^2 + \lambda = \lambda(\lambda + 1). \end{aligned}$$

The eigenvalues are $\lambda = 0$ and $\lambda = -1$.

(b) For each eigenvalue of A , find an eigenvector.

First $\lambda = -1$. We want something in the nullspace of $A - \lambda I = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$. The vector $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ will do the trick.

Now $\lambda = 0$. We want something in the nullspace of $A - \lambda I = A$. After row reduction (or just using the trick from class), one such vector is $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(c) Find matrices P and D where D is diagonal so that $A = PDP^{-1}$ (in other words, diagonalize A).

Our answers from above show that

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(d) Use your answer to (c) to compute A^6 .

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1}.$$

You could multiply this out and get the right answer. The shortcut is to notice that $D^6 = -D$, so $PD^6P^{-1} = -(PDP^{-1}) = -A$, and so

$$A^6 = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}.$$