

MATH 310, REVIEW SHEET 1

These notes are a very short summary of the key topics in the book (and follow the book pretty closely). You should be familiar with everything on here, but it's not comprehensive, so please be sure to look at the book and the lecture notes as well.

These notes also don't contain any examples at all. If you run into something unfamiliar, look at the notes or the book for an example! The goal of this sheet is just to remind you of what the major topics are.

I've also indicated some of the important "problem types" we've encountered so far and that you should definitely be able to do. There will inevitably be a problem on the exam not of one of the types listed.

1. LINEAR EQUATIONS

1.1. Systems of linear equations.

1.2. Row reduction and echelon forms. A linear equation is one of the form $a_1x_1 + \cdots + a_nx_n = b$, where the a_i 's and b 's are a bunch of numbers. A system of linear equations is a bunch of equations of this form, involving the same variables x_i . The goal is to find all the possible x_i that make the equations hold. In general, the number of solutions will be 0, 1, or infinite.

Given a linear system, the first step in solving it is usually to form the *augmented matrix* of the system. We then perform "elementary row operations" on the matrix to find the general solution. These row operations just correspond to simple manipulations of the equations.

There is basically a two-part process for finding all solutions to a linear system : first, use row operations to put the matrix in a standard form, "reduced row echelon form"; second, read off the general solution from the reduced row echelon form of the matrix.

There are three legal row operations:

- (1) Replace one row by the sum of itself and a multiple of another row;
- (2) Interchange two rows;
- (3) Multiply all entries in a row by a nonzero constant.

A matrix is in echelon form if:

- (1) All nonzero rows are above any rows of all zeros.
- (2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- (3) All entries in a column below a leading entry are zeros.

If the matrix additionally satisfies the following two requirements, it is in reduced echelon form:

- (1) The leading entry in each nonzero row is 1;
- (2) Each leading 1 is the only nonzero entry in its column.

These conditions may sound totally arbitrary; the point is that once a matrix is in reduced echelon form, there is no additional simplification to be done using algebraic operations. It's time to read off the solutions.

There is a five-step process for putting a matrix in reduced echelon form:

- (1) Begin with the leftmost nonzero column.
- (2) Select a nonzero entry in this column as a pivot. Usually you can just use the top entry; if not, swap to rows to move something nonzero to the top.
- (3) Create zeros below the pivot by adding multiples of the top row to subsequent rows.
- (4) Cover up the top nonzero row, and repeat the process on what remains.

Please, please, please practice some examples of row reduction. This technique is a little painful, but it is the basis for almost everything else in this chapter.

Once you reach reduced echelon form (or in fact echelon form), you should identify the pivots of your matrix: these are the entries in the matrix where which are the leading 1s in the reduced echelon form.

Once you've reached echelon form, you want to write down the general solution to the system of equations. The first thing to check is whether the system is consistent, i.e. whether there are any solutions at all. This is easy: if there's a row $[0 \ 0 \ 0 \mid b]$, where b is not 0, then there are no solutions (that's because a row of this kind corresponds to the linear equation $0 = b$).

If there's no row like this, then there are solutions: the system is consistent. Every variable that's not a basic variable is a "free variable", and can take any value. The basic variables are then determined by the values of the free variables, using the equations determined by the rows of the row reduced echelon form. The number of solutions is 1 if there are no free variables, and infinite if there are free variables (because you can plug in any number you want for the free variables.)

Here's another tidbit that can save time on exam: if you want to know whether there are free variables or not, it's enough to put the matrix into echelon form and see what the pivots are: you don't have to go the full way into reduced echelon form. If a question asks whether such-and-such has infinitely many solutions or only one (and doesn't ask you to actually find the solution), this is all you need to know. However, if you actually need to find the general solution, you're probably going to want to put it into rref anyway.

- Use row reduction to matrix a matrix in reduced echelon form
- Write down the general solution to a system of equations
- Determine if a system of equations is consistent
- Variant: for what values of a parameter h is a system consistent

1.3. Vector equations. This section introduces vectors and the rules for working with them. You should know what a vector is, how to add vectors, and how to draw vectors in \mathbb{R}^2 .

The most important topic here is linear combinations. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are a bunch of vectors, then $\mathbf{y} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ is called a *linear combination* of the \mathbf{v}_i .

The set of all linear combinations of a bunch of vectors is called the *span*. It's important to think of the span geometrically: if you have two vectors \mathbf{v}_1 and \mathbf{v}_2 in \mathbb{R}^3 , the span is the set of all the vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This span is a 2D plane in 3D space.

We introduced the notion of a *vector equation*: if $\mathbf{a}_1, \dots, \mathbf{a}_n$ and \mathbf{b} are a bunch of vectors, we can look at the vector equation $x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n = \mathbf{b}$: we are asking whether there exist

x_i 's such that \mathbf{b} is a linear combination of the \mathbf{a}_i 's, with weights x_i . In other words, the vector equation asks whether \mathbf{b} is a linear combination of the \mathbf{a}_i .

The vector equation is actually equivalent to a linear system with augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \mid b]$. So to solve it, you use exactly the same methods as before: the difference is that we're now thinking of the equation in terms of linear combinations.

- Do basic operations on vectors
- Convert a system of linear equations into a vector equation, and vice versa
- Find the general solution of a system of vector equations
- Determine if a vector is a linear combination of other given vectors

1.4. Matrix equations. Now we introduce matrices. If A is an $m \times n$ matrix and \mathbf{x} is a size- n vector, we can form the product $A\mathbf{x}$, which is a size- m vector. To get the vector you take a linear combination of the columns of A , with weights the entries in the vector \mathbf{x} : if the columns of A are called $\mathbf{a}_1, \dots, \mathbf{a}_n$, the product is $x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n$. There is also another way to compute the product, which is a little faster when doing it by hand, but amounts to basically the same thing. This is the “row–vector rule”, which you can read about on page 38 of the book.

Given a matrix A and a vector \mathbf{b} , one can ask whether $A\mathbf{x} = \mathbf{b}$ has any solutions. Since $A\mathbf{x}$ is a combination of the columns of A , this really amounts to asking whether \mathbf{b} can be written as a combination of the columns of A : solving for the vector \mathbf{x} just means that we are trying to find weights to make this work. So $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

The important thing here is that we now have three different ways to write a system of linear equations: as a bunch of equations involving the individual variables, as a vector equation, and as a matrix equation. They all amount to the same thing, and you solve them all the same way: using row reduction. You can see this worked out in an example in the lecture notes from 9/2.

Another useful fact is “Theorem 4” on page 37 of the book: given a matrix A , you can ask whether $A\mathbf{x} = \mathbf{b}$ has a solution for *every* \mathbf{b} . For some matrices A this is the case, while for other matrices A there are some \mathbf{b} 's for which there are no solutions. The answer is that $A\mathbf{x} = \mathbf{b}$ has a solution for every \mathbf{b} if the columns of A span \mathbb{R}^m , or equivalently, if A has a pivot position in every row.

- Multiply matrix \times vector
- Convert a system of linear equations or vector equation into a matrix equation
- Find the general solution of a matrix equation
- Check whether $A\mathbf{x} = \mathbf{b}$ has a solution for every \mathbf{b}

1.5. Solution sets of linear systems. The idea of this section is to think a bit about what the solution set of a system of linear equations is really like: although there can be infinitely many solutions, they can easily be parametrized.

A system is called *homogeneous* if it looks like $A\mathbf{x} = \mathbf{0}$. This corresponds to a system of linear equations where we are asking that various combinations of the variables are all 0. A homogeneous linear system is always consistent, since $\mathbf{x} = \mathbf{0}$ is a solution. It has a solution other than $\mathbf{x} = \mathbf{0}$ if and only if there is at least one free variable when you do row reduction to echelon form.

In general, given a linear system $A\mathbf{x} = \mathbf{0}$, you can write the solution in *parametric vector form*, which looks like $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, where \mathbf{u} and \mathbf{v} are two vectors. Here s and t are

parameters; in general the number of parameters you need is equal to the number of free variables. Make sure you know how to figure out the parametric vector form of the general solution.

This only handles homogeneous equations, but it can be adapted to inhomogeneous ones too. If you're looking at $A\mathbf{x} = \mathbf{b}$, you can find the general solution by finding the general solution to $A\mathbf{x} = \mathbf{0}$, in parametric vector form, and finding a single solution to $A\mathbf{x} = \mathbf{b}$ (e.g. by plugging in 0 for all the free variables, or just guessing). Then the general solution is the sum of these two things.

- Write solutions to a system in parametric vector form
- Describe geometrically the solution set of a system, using parametric vector form
- Find the general solution to an homogeneous linear system, given the general solution of the corresponding homogeneous linear system

1.6. Applications of linear systems. We only talked about two of the types of problems in this section, and only those are fair game: balancing a chemical reaction, and dealing with networks.

Balancing a chemical reaction is pretty straightforward: you are given a reaction with reactants on one side, and products on the other. The challenge is to find how many molecules of each of the reactants and products are needed to balance the equation. To do this, you translate it to a linear system: give a variable name x_i to the coefficient in front of each of the reactants and products. For each of the elements that appears in the reaction, you get a linear equation: the number of atoms of the element appearing on the left is equal to the number of atoms on the right. Then you solve the linear system.

The other thing we talked about was network flow. Here's the set up: you're given a network, which consists of a bunch of nodes, and a bunch of branches connecting the nodes. You can think of it as a network of streets, where the nodes are intersections, and the branches are the roads between the intersections. Given the flow along some of the branches, the goal is to figure out what the flow must be along the others, where it is not known. The strategy is similar: for each of the branches where you don't know the flow, give a variable name to it. Now you get a bunch of linear equations: at each node, the total incoming flow must equal the total outgoing traffic flow. Then you get one extra equation: the total flow into the network must equal the total flow out of the network. This is a bunch of linear equations: now set up the augmented matrix and solve by row reduction.

- Balance a chemical equation by setting up a linear system
- Find the flows in a network by setting up a linear system

1.7. Linear independence. Next up is linear independence. Say you have a bunch of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. They're *linearly independent* if the only way to get $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$ is if all the x_i are 0.

Checking is easy: we want to know if the vector equation $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$ has any solutions other than 0. This is something that you can check by row reduction: it's a homogenous linear system, so set up an augmented matrix with a column of 0's, go through row reduction, and check.

For two vectors to be linearly independent means that they are not collinear (this you can check just by looking at them: is one a multiple of the other?); for three vectors to be linearly independent means that they are not all in the same plane.

- Check whether a set of vectors is linearly independent

- Find values of a parameter h for which a set of vectors is linearly independent

1.8. Introduction to linear transformations.

1.9. The matrix of a linear transformation. If you have a $m \times n$ matrix A , you can think of it as defining a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$: it's a rule that takes as input a size n vector, and gives as output a size m vector.

You can often think about linear transformations graphically: draw a few vectors, and then draw the new vectors you get after applying T . This can help you work figure out a simple geometric description of what a linear transformation does.

Officially, a transformation T is linear if $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$. This roughly means that it sends straight lines to other straight lines, and sends $\mathbf{0}$ to $\mathbf{0}$. Any transformation that's given by a matrix is linear, and it turns out that the opposite is true as well: any transformation that's linear is determined by some matrix.

An important task is to be able to figure out the matrix that gives a linear transformation. For example, the transformation might be described to you geometrically or by some equations, and your task is to write down the matrix that does the transformation in question. There is a recipe for this. First, start with the vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ (you'll need more vectors if your transformation is between vectors with more entries). Figure out what $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are: to do this, use the description of T that's given to you to work out where T sends \mathbf{e}_1 and \mathbf{e}_2 . This could involve doing some geometry, or plugging in $(1, 0)$ to some equation.

Once you've done that, form a matrix A whose columns are $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, etc. That's the matrix for your transformation.

There are also two new definitions introduced in this chapter: if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, given by a matrix A , then:

- (1) T is onto if each \mathbf{b} in \mathbb{R}^m is the image of at least one \mathbf{x} . To check this, put the matrix A into rref, and check if there is a pivot in every row (if yes, then T is onto).
- (2) T is one-to-one if each \mathbf{b} in \mathbb{R}^m is the image of at most one \mathbf{x} in \mathbb{R}^n . This is equivalent to T having linearly independent columns: you can check that by our strategy for checking linear independence given above.

- Find the image of a vector under a linear transformation
- Check whether a vector is in the range of a linear transformation
- Describe a linear transformation geometrically
- Find the matrix for a linear transformation, given a description in some other form
- Check whether a transformation is one-to-one and/or onto

1.10. More linear models. Electrical networks: given a diagram of a simple circuit containing resistors and batteries, we can find the current through each branch of the circuit. To do this, think of the circuit as a bunch of loops, with some current I_j flowing through it. Our goal is to solve for the I_j 's. From each loop, you get a linear equation, using Kirchhoff's voltage law: if you add up the voltage drops RI for each resistor, it's equal to the sum of the batteries in the loop. This gives a system of linear equations that you can solve by row reduction. (This explanation might not make so much sense without an example; take a look at Example 2 on page 83.)

2. MATRIX ALGEBRA

2.1. Matrix operations. You should know how to perform the following operations on a matrix: sum, scalar multiple, multiplication, transpose, powers. This is mostly a matter of practice: make sure you do some of each.

The interesting one is multiplying matrices; the rest work about like you'd expect. Here's the official definition: given two matrices A and B , first check if the number of columns of A is equal to the number of rows of B ; if not, then AB isn't defined. If yes, then AB is given by $A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, where these are the columns of B .

There's another way to multiply matrices that's a little faster when you're doing it by hand: this is the "row-column method" which you can read about in the notes or the book.

Important to remember is that multiplication of matrices doesn't work as nicely as multiplication of numbers. For example, AB and BA aren't always the same (even when both are defined). It's also not true that if $AB = AC$, then $B = C$: you can't "divide both sides of the equation by A ".

- Perform basic matrix operations
- Understand the basic properties of these operations

2.2. The inverse of a matrix. If A is an $n \times n$ matrix (note: it must be square!), we say A is invertible if there is another $n \times n$ matrix C such that $AC = CA = I_n$, where I_n is the identity matrix. In this case we write $C = A^{-1}$.

One use of this is that if you're trying to solve $A\mathbf{x} = \mathbf{b}$, and A is invertible, there is a unique solution, and it's given by $\mathbf{x} = A^{-1}\mathbf{b}$. This method is slow to actual solve equations, but it's helpful when you're trying to derive formulas for things.

For a 2×2 matrix, there's a formula for the inverse:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Unless $ad - bc = 0$: then the matrix isn't invertible.) For bigger matrices, the best way to find the inverse is this (assuming Matlab is not available to you...): write down the $n \times 2n$ augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$, run row reduction until you reach $\begin{bmatrix} I_n & B \end{bmatrix}$. The matrix B that appears on the right is the inverse.

- Find the inverse of a 2×2 matrix
- Find the inverse of a bigger matrix
- Solve $A\mathbf{x} = \mathbf{b}$ when A is invertible, using matrix inversion

2.3. Characterizations of invertible matrices. This is a weird chapter, in that there is only one thing in it. However, it's an important thing: if A is a square matrix, the following statements are either all true or all false.

- (1) A is invertible
- (2) Row reduction on A ends up with I_n
- (3) A has n pivots (which must be on the diagonal)
- (4) $A\mathbf{x} = 0$ has only the trivial solution
- (5) The columns of A are linearly independent
- (6) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
- (7) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
- (8) The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every \mathbf{b}

- (9) The columns of A span \mathbb{R}^n
- (10) There is a matrix C with $AC = I_n$
- (11) There is a matrix D with $DA = I_n$
- (12) A^T is invertible
- (13) (For 2×2 only, for now) $\det A = ad - bc \neq 0$

The upshot is that to check if A is invertible, you can check any of these other things using the method of your choice. Note that C and D in (10) and (11) end up being the same: the interesting thing is that if you can find a C with $AC = I_n$, then it's automatically true that $CA = I_n$ too.

- Given a matrix, figure out whether it's invertible