Math 310, Lesieutre
Problem set #11
November 11, 2015

Problems for M 11/2:

5.4.11 Let $\mathcal{B}$ be the basis given by

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the $\mathcal{B}$-matrix for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$$

(This just means the matrix for the transformation T, but where we use the basis $\mathcal{B}$ on both sides.)

5.4.13 Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ with respect to which the transformation is diagonal.

1. Early in the course I mentioned that linear algebra gives you a way to find a formula for the Fibonacci numbers. I didn't get to it in lecture, so I will let you work it out. The Fibonacci numbers are a sequence defined by $F_1 = 1$ and $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$. They start off $1, 1, 2, 3, 5, 8, 13, \ldots$. You can read lots of fun facts about them on Wikipedia.

   (a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{x}$ be the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $A\mathbf{x}$, $A^2\mathbf{x}$, $A^3\mathbf{x}$. Convince yourself that $A^n\mathbf{x}$ is the vector $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$.

   (b) Computing $A^n\mathbf{x}$ directly is hard, but we can do it by working with coordinates in an eigenbasis. First, find the eigenvectors and eigenvalues of A (hint: your answer will be a little messy, involving some $\sqrt{5}$'s.) Then write down a diagonalization of A.

   (c) Let $\mathcal{B}$ be the basis given by the eigenvectors you found. Compute the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

   (d) Our formula for linear transformations in an eigenbasis tells us that $[A^n(\mathbf{x})]_{\mathcal{B}} = D^n[\mathbf{x}]_{\mathcal{B}}$, where D is the diagonalization of A. Use your answers to the previous questions to find $[A^n(\mathbf{x})]_{\mathcal{B}}$.

(e) Convert this back into regular coordinates to get an expression for $A^n(\mathbf{x})$. What is your formula for F_{n+1}?

Problems for W 11/4:

5.4.18 Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 3×3 matrix with eigenvalues 5 and -2. Does there exist a basis $\mathcal{B}$ such that the $\mathcal{B}$-matrix for T is a diagonal matrix? Discuss.

1. Find the roots of $x^2 - 4x + 13 = 0$.
2. Find $(3 + 4i)(2 - 6i)$.
3. Find $\frac{3+4i}{2-6i}$.
4. Write $1 + i$ in polar form, $re^{i\theta}$. Use this to compute $(1 + i)^5$.

Problems for F 11/6:

5.5.1 Find the (possibly complex) eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}.$$

5.5.4 Ditto, with

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}.$$

5.5.9 Find the eigenvalues of A. The transformation determined by A is a composition of a rotation and a scaling; give the angle of the rotation, and the scaling factor. (Hint: look at Example 6).

$$A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

5.5.13 Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (both with real entries) such that $A = PCP^{-1}$. You might find your answer to the first question useful.