Math 310, Lesieutre
Problem set #10
November 4, 2015

Problems for M 10/26:

5.1.1 *Is $\lambda = 2$ an eigenvalue of*

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}?$$

Why or why not?

We have $A - 2I = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The determinant is 0, which means that $A - 2I$ has a nullspace, and so there is an eigenvector with eigenvalue 2. (We aren't asked to actually figure out what the eigenvector is.)

5.1.2 *Is $\lambda = -2$ an eigenvalue of*

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}?$$

Why or why not?

For this one, we have $A - \lambda I = A + 2I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. Again the determinant is 0, which tells us that there is a nullspace, which tells us that it is an eigenvector.

5.1.15 *Find a basis for the eigenspace corresponding to each listed eigenvalue.*

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \quad \lambda = 3.$$

The eigenspace is the nullspace of $A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$.

We notice right away that all the rows are multiples of each other. That means that the rank is 1, so we're expecting the dimension of the nullspace to be 2. To actually find the eigenvalues, use row reduction:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Ther are two free variables, x_2 and x_3. Let s and t be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the nullspace of $A - 3I$, and hence a basis for the eigenspace of A corresponding to eigenvalue 3, is given by the two vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

5.1.18 *Find the eigenvalues of the matrix*

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}.$$

This is a lower-triangular matrix, so we can just read off the diagonal entries: the eigenvalues are 4, 0, and -3.

5.1.19 *For*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix},$$

find one eigenvalue, with no calculation. Justify your answer.

The rows are all the same, which means the rank is 1 (since the row space is one-dimensional). That in turn means that the nullspace is 2-dimensional. So $\lambda = 0$ is an eigenvalue! (For this purpose it's irrelevant that the eigenspace has dimension 2 – just that it's not 0.)

Problems for W 10/28:

5.2.1 *Find the characteristic polynomial and the eigenvalues of the matrix*

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}.$$

We have

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}.$$

The determinant is $(2-\lambda)(2-\lambda) - 49 = 4 - 4\lambda + \lambda^2 - 49 = \lambda^2 - 4\lambda - 45$. This factors as $(\lambda - 9)(\lambda + 5)$ (use the quadratic formula if you need to), and so the eigenvalues are 9 and -5.

5.2.5 *Find the characteristic polynomial and the eigenvalues of the matrix*

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}.$$

This time

$$\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} = 8 - 6\lambda + \lambda^2 - (-1) = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2.$$

So the eigenvalues are 3 and 3.

5.2.9 *Find the characteristic polynomial and and eigenvalues of the matrix. (Hint: it's 3×3, so your best bet is to use cofactor expansion or the special 3×3 formula.*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}.$$

We have

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & -\lambda \end{bmatrix}$$

Let's do it with the formula for 3×3 determinants: we get

$$(1-\lambda)(3-\lambda)(-\lambda) + (-1)(2)(6) - (1-\lambda)(-1)(6) = \lambda^3 + 4\lambda^2 - 9\lambda - 6.$$

Finding the eigenvalues of this would be a mess; your best approach would be to do it approximately using Newton's method. This polynomial doesn't factor in any reasonable way.

5.2.12 *Find the characteristic polynomial and and eigenvalues of the matrix (I know this is getting a little repetitive, but this is a very important thing to be able to do!)*

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

We want

$$\det(A-\lambda) = \det A = \begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}.$$

Expanding by cofactors in row 3, this is

$$(2-\lambda)\det \begin{bmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda)(4-\lambda).$$

The eigenvalues are 2, -1, and 4.

5.2.15 *Find the eigenvalues, repeated according to their multiplicities (i.e. if an egienvalue is a double root of the characteristic polynomial, list it twice.)*

$$A = \begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It's triangular, so this is easy: read them off the diagonal. The eigenvalues of 4, 3, 3, 1.

Problems for F 10/30:

5.3.1 *Let $A = PDP^{-1}$, where P and D are the matrices listed below. Compute A^4.*

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

By the formula for raising PDP^{-1} to a power, this is

$$A^4 = (PDP^{-1})^4 = PD^4P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}.$$

5.3.3 *Let*

$$A = \begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix}.$$

Use the factorization

$$\begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

to compute A^k, where k is any positive integer (your answer should be in terms of a, b, and k).

We have

$$A^k = PD^kP^{-1} = \begin{bmatrix} a & 0 \\ 3(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a^k & 0 \\ 3a^k & b^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} a^k & 0 \\ 3a^k - 3b^k & b^k \end{bmatrix}.$$

As a sanity check, you can immediately see that this gives the right answer for $k = 0$ and $k = 1$.

5.3.7 *Diagonalize the matrix*

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}.$$

The eigenvalues are 1 and -1 (it's triangular, so this doesn't require any work). For $\lambda = 1$, we have $A - 1I = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$, and so $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector. For $\lambda = -1$, we get $A - \lambda I = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$, for which an eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5.3.11 *Same deal:*

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Hint: the eigenvalues of A are 1, 2, and 3. (But you still have to find the eigenvectors yourself.)

We have to find an eigenvector for each eigenvalue:

$$A - 1I = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$A - 2I = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix},$$

$$A - 3I = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix},$$

So it is diagonalized as $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$