Math 310, Lesieutre
Problem set #11
November 11, 2015

Problems for M 11/2:

5.4.11 *Let $\mathcal{B}$ be the basis given by*

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the $\mathcal{B}$-matrix for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$$

(This just means the matrix for the transformation T, but where we use the basis $\mathcal{B}$ on both sides.)

We have a formula to find the matrix for a transformation in a given basis. It's

$$M = [[T(\mathbf{b}_1)]_{\mathcal{B}} \; [T(\mathbf{b}_2)]_{\mathcal{B}}].$$

For us,

$$T(\mathbf{b}_1) = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$T(\mathbf{b}_2) = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

But to get M, we need to know the $\mathcal{B}$-coordinates of both of these vectors. For that, we need to use our older formulas:

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}^{-1}(T(\mathbf{b}_1))$$
$$[T(\mathbf{b}_2)]_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}^{-1}(T(\mathbf{b}_2))$$

We have

$$\mathcal{P}_{\mathcal{B}}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

We then obtain

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$[T(\mathbf{b}_2)]_{\mathcal{B}} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

So at the end of the day, the $\mathcal{B}$-matrix is given by

$$M = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}.$$

5.4.13 *Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ with respect to which the transformation is diagonal.

A matrix becomes diagonal when you work in an eigenbasis (i.e. a basis made up of eigenvectors. So we just have to find the eigenvectors and that'll be the basis we want.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

For $\lambda = 1$, we have

$$A - \lambda I = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix},$$

and so the eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Likewise for $\lambda = 3$, it's

$$A - \lambda I = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix},$$

and the eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. So our basis $\mathcal{B}$ has two vectors,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

1. *Early in the course I mentioned that linear algebra gives you a way to find a formula for the Fibonacci numbers. I didn't get to it in lecture, so I will let you work it out. The Fibonacci numbers are a sequence defined by $F_1 = 1$ and $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$. They start off $1, 1, 2, 3, 5, 8, 13, \ldots$. You can read lots of fun facts about them on Wikipedia.*

   (a) *Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{x}$ be the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $A\mathbf{x}$, $A^2\mathbf{x}$, $A^3\mathbf{x}$. Convince yourself that $A^n\mathbf{x}$ is the vector $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$.*

   We have

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ A^2\mathbf{x} = A(A\mathbf{x}) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ A^3\mathbf{x} = A(A\mathbf{x}) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}. \end{aligned}$$

   It seems to work that $A^n\mathbf{x} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$; every time we apply A, the new first entry is the sum of the preceding ones, and the new second entry is the old first entry.

(b) *Computing $A^n\mathbf{x}$ directly is hard, but we can do it by working with coordinates in an eigenbasis. First, find the eigenvectors and eigenvalues of A (hint: your answer will be a little messy, involving some $\sqrt{5}$'s.) Then write down a diagonalization of A.*

The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1.$$

To solve these we need the quadratic formula.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{5}}{2}.$$

This number $\frac{1+\sqrt{5}}{2}$ is going to show up a lot in this problem, so let's give it a name: ϕ (better known as the golden ratio). For the eigenvalue with a $-$ instead of a plus, write μ.

We know find the eigenvectors:

$$A - \phi I = \begin{bmatrix} 1-\phi & 1 \\ 1 & -\phi \end{bmatrix}.$$

Using the trick for 2×2 eigenvectors, our eigenvector is $\begin{bmatrix} \phi \\ 1 \end{bmatrix}$ (I'm using the second row instead of the first here for simplicity; you'll get the same answer either way eventually.)

The other eigenvector is similarly found via

$$A - \mu I = \begin{bmatrix} 1-\mu & 1 \\ 1 & -\mu \end{bmatrix},$$

whence we get $\begin{bmatrix} \mu \\ 1 \end{bmatrix}$.

So the diagonalization is $A = PDP^{-1}$, where

$$P = \begin{bmatrix} \phi & \mu \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \phi & 0 \\ 0 & \mu \end{bmatrix}.$$

(c) *Let $\mathcal{B}$ be the basis given by the eigenvectors you found. Compute the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.*

To get the coordinate vector we need

$$\begin{aligned} [\mathbf{x}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}^{-1}\mathbf{x} &= \begin{bmatrix} \phi & \mu \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\phi - \mu} \begin{bmatrix} 1 & -\mu \\ -1 & \phi \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\phi - \mu} \begin{bmatrix} 1-\mu \\ \phi - 1 \end{bmatrix}. \end{aligned}$$

Notice that

$$\phi - \mu = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}.$$

So

$$[\mathbf{x}]_{\mathcal{B}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1-\mu \\ \phi - 1 \end{bmatrix}.$$

(d) *Our formula for linear transformations in an eigenbasis tells us that* $[A^n(\mathbf{x})]_{\mathcal{B}} = D^n[\mathbf{x}]_{\mathcal{B}}$*, where* D *is the diagonalization of* A*. Use your answers to the previous questions to find* $[A^n(\mathbf{x})]_{\mathcal{B}}$*.*

Now,

$$\begin{aligned}[A^n(\mathbf{x})]_{\mathcal{B}} = D^n[\mathbf{x}]_{\mathcal{B}} &= \begin{bmatrix} \phi & 0 \\ 0 & \mu \end{bmatrix}^n \frac{1}{\sqrt{5}} \begin{bmatrix} 1-\mu \\ \phi-1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \phi^n & 0 \\ 0 & \mu^n \end{bmatrix} \begin{bmatrix} 1-\mu \\ \phi-1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \phi^n - \mu\phi^n \\ \phi\mu^n - \mu^n \end{bmatrix}\end{aligned}$$

But

$$\phi\mu = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{-4}{4} = -1.$$

Then $\mu\phi^n = (\mu\phi)\phi^{n-1} = -\phi^{n-1}$, and $\phi\mu^n = -\mu^{n-1}$, so

$$[A^n(\mathbf{x})]_{\mathcal{B}} = \frac{1}{\sqrt{5}} \begin{bmatrix} \phi^n + \phi^{n-1} \\ -\mu^{n-1} - \mu^n \end{bmatrix}.$$

(e) *Convert this back into regular coordinates to get an expression for* $A^n(\mathbf{x})$*. What is your formula for* F_{n+1}*?*

Now we change coordinates back to the usual ones.

$$A^n(\mathbf{x}) = \mathcal{P}_{\mathcal{B}}[A^n(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} \phi & \mu \\ 1 & 1 \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} \phi^n + \phi^{n-1} \\ -\mu^{n-1} - \mu^n \end{bmatrix} \right) = \frac{1}{\sqrt{5}} \begin{bmatrix} \phi^{n+1} + \phi^n - \mu^{n+1} - \mu^n \\ \phi^n + \phi^{n-1} - \mu^n - \mu^{n-1} \end{bmatrix}$$

So our answer is that

$$F_{n+1} = \frac{1}{\sqrt{5}} \left(\phi^n + \phi^{n-1} - \mu^n - \mu^{n-1} \right).$$

Equivalently,

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^{n-1} + \phi^{n-2} - \mu^{n-1} - \mu^{n-2} \right).$$

This can be simplified a little bit. We have $\phi^{n-1} + \phi^{n-2} = \phi^{n-2}(1+\phi) = \phi^{n-2}(\phi^2) = \phi^n$, using the fact that $\phi^2 - \phi - 1 = 0$. Similarly $\mu^{n-1} + \mu^{n-2} = \mu^n$. So

$$F_n = \frac{\phi^n - \mu^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Notice that $\left|(1-\sqrt{5})/2\right| < 1$, and so when n is very big this is roughly just $\phi^n/\sqrt{5}$. This explains why the ratio of consecutive Fibonacci numbers is close to the golden ratio: it's an eigenvalue of the matrix that generates them!

Problems for W 11/4:

5.4.18 *Define* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $T(\mathbf{x}) = A\mathbf{x}$*, where* A *is a* 3×3 *matrix with eigenvalues* 5 *and* -2*. Does there exist a basis* $\mathcal{B}$ *such that the* $\mathcal{B}$*-matrix for* T *is a diagonal matrix? Discuss.*

There's a matrix with respect to which the transformation is diagonal if the matrix A can be diagonalized. They haven't told us much about A. Because it's a 3×3 matrix with only two eigenvalues, one of them has to be repeated. So maybe A is diagonalizable and maybe it isn't; it depends on whether or not the eigenspace corresponding to this eigenvalue is 2-dimensional or only 1-dimensional.

1. *Find the roots of* $x^2 - 4x + 13 = 0$.

   Use the quadratic formula:

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i.$$

2. *Find* $(3 + 4i)(2 - 6i)$.

   Just multiply it out:

$$(3-4i)(2-6i) = (3)(2)+(3)(-6i)+(-4i)(2)+(-4i)(-6i) = 6-18i-8i-24 = -18-26i.$$

3. *Find* $\frac{3+4i}{2-6i}$.

   For this one you want to do:

$$\frac{3+4i}{2-6i} = \frac{3+4i}{2-6i}\frac{2+6i}{2+6i} = \frac{-18+26i}{40} = \frac{-9+13i}{20}.$$

4. *Write* $1 + i$ *in polar form,* $re^{i\theta}$*. Use this to compute* $(1 + i)^5$.

   In polar form we have

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}.$$

   Then

$$(1+i)^5 = (\sqrt{2}e^{i\pi/4})^5 = (\sqrt{2})^5 e^{5\pi i/4} = 4\sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$$
$$= 4\sqrt{2}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = -4 - 4i.$$

Problems for F 11/6:

5.5.1 *Find the (possibly complex) eigenvalues and eigenvectors for the matrix*

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}.$$

For this one we have

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = 3 - 4\lambda + \lambda^2 - (-2) = \lambda^2 - 4\lambda + 5.$$

The quadratic formula gives us

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i.$$

Let's use $2 - i$. The complex eigenvector is found by computing

$$A - (2-i)I = \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}$$

The eigenvector is

$$\mathbf{v} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

There are lots of other possible answers; just multiply both entries of this one by any complex number you want.

The eigenvector for $2 + i$ is the complex conjugate of this one, namely

$$\mathbf{w} = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}.$$

5.5.4 *Ditto, with*

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}.$$

This time we get

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = 15 - 8\lambda + \lambda^2 - (-2) = \lambda^2 - 8\lambda + 17.$$

Let's do this one without the quadratic formula. Completing the square, this is

$$(\lambda - 4)^2 + 1,$$

which is going to be 0 when $\lambda = 4 \pm i$.

Let's find the eigenvector for $4 - i$. We have

$$A - \lambda I = \begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}.$$

By the "switch the two things in the first row and add a minus sign" trick we've discussed, this gives

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}.$$

The eigenvector for the eigenvalue $4+i$ is just the complex conjugate of this, which is

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}.$$

5.5.9 *Find the eigenvalues of A. The transformation determined by A is a composition of a rotation and a scaling; give the angle of the rotation, and the scaling factor. (Hint: look at Example 6).*

$$A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

Let's follow the hint. Example 6 tells us that a matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ has eigenvalues $a \pm bi$. Our matrix is of this form, with $a = -\sqrt{3}/2$ and $b = -1/2$. This has $r = \sqrt{a^2+b^2} = 1$ and $\theta = 7\pi/6$, so it corresponds to a rotation by $7\pi/6$ and a scaling by a factor of 1 (the latter of which doesn't actually do anything).

5.5.13 *Consider the matrix*

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (both with real entries) such that $A = PCP^{-1}$. You might find your answer to the first question useful.

We have a recipe for this. We already found the eigenvalue $2-i$ with eigenvector $\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$. Then $A = PCP^{-1}$, where

$$P = [\mathrm{Re}(\mathbf{v})\ \mathrm{Im}(\mathbf{v})], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

For our matrix, this is

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

If you multiply out PCP^{-1}, you'll find that this indeed does the trick.