Math 310, Lesieutre
Problem set #12
November 18, 2015

Problems for M 11/9:

5.7.1 *A particle moving in a force field has a position vector* $\mathbf{x}$ *that satisfies* $\mathbf{x}' = A\mathbf{x}$. *The* 2×2 *matrix* A *has eigenvalues* 4 *and* 2, *with corresponding eigenvectors* $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. *Find the position of the particle at time* t, *assuming that* $\mathbf{x}(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

All the solutions are obtained as linear combinations of the solutions $\mathbf{v}e^{\lambda t}$, so we want

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$$

When we plug in $t = 0$, we get

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

This is solved by $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -3/2 \end{bmatrix}$. So the solution we want to the differential equation is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{5}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}.$$

5.7.7 *Find a change of variable that decouples the equation* $\mathbf{x}' = A\mathbf{x}$. *Write the equation* $\mathbf{x}(t) = P\mathbf{y}(t)$ *and show the calculation that leads to the uncoupled system* $\mathbf{y}' = D\mathbf{y}$, *specifying* P *and* D.

$$A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$$

We want to make the change of variables $\mathbf{y} = P^{-1}\mathbf{x}$. The usual calculation shows that the eigenvalues are $\lambda = 4$ and $\lambda = 6$, with eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively, so

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

Note $P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}$.

So we want $y_1 = -1/2x_1 + 1/2x_2$ and $y_2 = 3/2x_1 - 1/2x_2$. Then

$$\begin{aligned}
y_1' &= -\frac{1}{2}x_1' + \frac{1}{2}x_2' \\
&= -\frac{1}{2}(7x_1 - x_2) + \frac{1}{2}(3x_1 + 3x_2) = -2x_1 + 2x_2 = 4y_1 \\
y_2' &= 3/2x_1' - 1/2x_2' = 3/2(7x_1 - x_2) - 1/2(3x_1 + 3x_2) = 9x_1 - 3x_2 = 6y_2.
\end{aligned}$$

The system has become decoupled, like we wanted.

You can also write this is matrix form, where it's a little shorter but less transparent what's going on. Take $\mathbf{y} = P^{-1}\mathbf{x}$, and

$$\mathbf{y}' = P^{-1}\mathbf{x}' = P^{-1}A\mathbf{x} = P^{-1}PDP^{-1}\mathbf{x} = DP^{-1}\mathbf{x} = D\mathbf{y}.$$

5.7.9 *Construct the general solution of* $\mathbf{x}' = A\mathbf{x}$ *involving complex eigenfunctions and then obtain the general real solution.*

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

The eigenvalues are $-2 + i$ and $-2 - i$, with eigenvectors $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.

The complex solutions are

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(-2-i)t}.$$

To get the real solutions, we pick one of these, and take the real and complex parts. I'll use the first one.

$$\begin{aligned}
\begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} &= \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2t}(\cos t + i\sin t) = e^{-2t}\begin{bmatrix} (\cos t + \sin t) + (-\cos t + \sin t)i \\ \cos t + (\sin t)i \end{bmatrix} \\
\text{Re} &= \begin{bmatrix} e^{-2t}(\cos t + \sin t) \\ e^{-2t}(\cos t) \end{bmatrix} \\
\text{Im} &= \begin{bmatrix} e^{-2t}(-\cos t + \sin t) \\ e^{-2t}\sin t \end{bmatrix}
\end{aligned}$$

The general real solution is given by all combinations of these two:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t}(\cos t + \sin t) \\ e^{-2t}(\cos t) \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t}(-\cos t + \sin t) \\ e^{-2t}\sin t \end{bmatrix}$$

Problems for F 11/13:

6.1.1 *Let* $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. *Compute* $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{u} \cdot \mathbf{v}$, *and* $\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{u} &= (-1)(-1) + (2)(2) = 5 \\ \mathbf{u} \cdot \mathbf{v} &= (-1)(4) + (2)(6) = 8 \\ \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} &= \frac{8}{5}. \end{aligned}$$

6.1.14 *Find the distance between the two vectors*

$$\begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}.$$

The distance between the two is the length of this vector, which is

$$\sqrt{(4)^2 + (-4)^2 + (-6)^2} = \sqrt{68}.$$

6.1.16 *Are the following two vectors orthogonal?*

$$\begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}.$$

The dot product is

$$\begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = (12)(2) + (3)(-3) + (-5)(3) = 0.$$

Yep, they're orthogonal.

6.1.26 *Let* $\mathbf{u} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, *and let* W *be the set of all vectors* $\mathbf{x}$ *with* $\mathbf{u} \cdot \mathbf{x} = 0$. *Explain why* W *is a subspace of* $\mathbb{R}^3$, *and describe it geometrically.*

This is the orthogonal complement of $V = \text{span}(\mathbf{u})$, the set of all multiples of $\mathbf{u}$. The orthogonal complement of a subspace is always a subspace.

Geometrically, V is the plane that's perpendicular to this vector.

Alternately, it's the nullspace of the matrix

$$A = \begin{bmatrix} 5 & -6 & 7 \end{bmatrix}.$$

The nullspace of any matrix is a subspace.