Math 310, Lesieutre
Problem set #13
November 25, 2015

Problems for M 11/16:

6.2.1 *Determine whether the following vectors are an orthogonal set:*

$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}.$$

We need to check each pair and see if they dot to 0:

$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = -5 + 8 - 3 = 0$$
$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = -3 - 16 + 21 = 2$$
$$\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = 15 - 8 - 7 = 0.$$

They're not orthogonal, since the first and third are not perpendicular to each other.

6.2.8 *Show that the following vectors are an orthogonal basis for* $\mathbb{R}^2$*, and express* $\mathbf{x}$ *as a linear combination of the* $\mathbf{u}$*'s.*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}.$$

We have

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 12 - 12 = 0.$$

So they're orthogonal. A set of orthogonal vectors is always linearly independent, so we don't need to check that by row reduction like we usually would (though there's no harm in doing so).

We then have

$$\mathbf{x} = \frac{\mathbf{u}_1 \cdot \mathbf{x}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{x}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$
$$= \frac{39}{13}\mathbf{u}_1 + \frac{26}{52}\mathbf{u}_2 = 3\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2.$$

6.2.10 *Show that the following vectors are an orthogonal basis for* $\mathbb{R}^3$*, and express* $\mathbf{x}$ *as a linear combination of the* $\mathbf{u}$*'s.*

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

We compute the dot products of all these things to show that they're an orthogonal set.

$$\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 0,$$

$$\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 0.$$

Looks good. Then

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{u}_1 \cdot \mathbf{x}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{x}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{u}_3 \cdot \mathbf{x}}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 \\ &= \frac{24}{18}\mathbf{u}_1 + \frac{3}{9}\mathbf{u}_2 + \frac{6}{18}\mathbf{u}_3 = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3. \end{aligned}$$

6.2.12 *Compute the orthogonal projection of* $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *onto the line through* $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ *and the origin.*

This is just going to be

$$\frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{-4}{10} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}.$$

6.2.13 *Let* $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$*. Write* $\mathbf{y}$ *as a sum of two orthogonal vectors, one in he span of* $\mathbf{u}$ *and one orthogonal to* $\mathbf{u}$*. (We didn't do one quite like this in lecture; take a look at Example 3 in the book.)*

We can do this like the above one: take the component that's parallel to $\mathbf{u}$. The result when we subtract is going to be perpendicular to $\mathbf{u}$ (this is how Gram–Schmidt works).

We get

$$\text{proj}_{\mathbf{y}}\, \mathbf{u} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{-13}{65}\mathbf{u} = -\frac{1}{5}\mathbf{u} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}.$$

Then the other part is

$$\mathbf{y} - \text{proj}_{\mathbf{y}}\,\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}.$$

We know that this should work, but let's do a sanity check here. Is this vector actually orthogonal to $\mathbf{u}$, like it's supposed to be?

$$\begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \frac{56}{5} - \frac{56}{5} = 0,$$

as it must.

Problems for W 11/20:

6.3.3 *Verify that the given vectors are an orthogonal set, and then find the projection of* $\mathbf{y}$ *onto* $W = span(\mathbf{u}_1, \mathbf{u}_2)$.

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

We have

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = (1)(-1) + (1)(1) + (0)(0) = 0.$$

So they're orthogonal.

The orthogonal projection is

$$\begin{aligned} \text{proj}_{\mathbf{u}_1,\mathbf{u}_2}\,\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\ &= \frac{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}. \end{aligned}$$

6.3.9 *Let* W *be the subspace spanned by* $\mathbf{u}_1$, $\mathbf{u}_2$ *and* $\mathbf{u}_3$. *Write* $\mathbf{y}$ *as the sum of a vector in* W *and a vector orthogonal to* W.

$$\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

This time we want

$$\begin{aligned}
\operatorname{proj}_{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\
&= \frac{\begin{bmatrix}4\\3\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}4\\3\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix}}{\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix}}\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix} + \frac{\begin{bmatrix}4\\3\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\\1\end{bmatrix}}{\begin{bmatrix}-1\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\\1\end{bmatrix}}\begin{bmatrix}-1\\0\\1\\1\end{bmatrix} \\
&= \frac{8}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \frac{6}{15}\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix} + \frac{0}{3}\begin{bmatrix}-1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{34}{15}\\\frac{58}{15}\\\frac{2}{5}\\\frac{28}{15}\end{bmatrix}.
\end{aligned}$$

6.3.12 *Find the closest point to* $\mathbf{y}$ *in the subspace spanned by* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*.*

$$\mathbf{y} = \begin{bmatrix}3\\-1\\1\\13\end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}-4\\1\\0\\3\end{bmatrix}.$$

This is the same general deal as the last one.

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}_1,\mathbf{v}_2}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 \\
&= \frac{\begin{bmatrix}3\\-1\\1\\13\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}}{\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \frac{\begin{bmatrix}3\\-1\\1\\13\end{bmatrix}\cdot\begin{bmatrix}-4\\1\\0\\3\end{bmatrix}}{\begin{bmatrix}-4\\1\\0\\3\end{bmatrix}\cdot\begin{bmatrix}-4\\1\\0\\3\end{bmatrix}}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} \\
&= \frac{30}{10}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \frac{26}{26}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}.
\end{aligned}$$

6.3.15 *Let*

$$\mathbf{y} = \begin{bmatrix}5\\-9\\5\end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix}-3\\-5\\1\end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix}-3\\2\\1\end{bmatrix}.$$

Find the distance from $\mathbf{y}$ *to the plane in* $\mathbb{R}^3$ *spanned by* $\mathbf{u}_1$ *and* $\mathbf{u}_2$*. (Hint: what point in that plane is closest to* $\mathbf{y}$*?)*

We're supposed to find the distance from $\mathbf{y}$ to the plane. To do that, we figure out what the closest point on the plane is (which we know how to do), and then we figure out the distance to that point.

Here we go again:

$$\begin{aligned}
\text{proj}_{\mathbf{u}_1,\mathbf{u}_2}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 \\
&= \frac{\begin{bmatrix}5\\-9\\5\end{bmatrix}\cdot\begin{bmatrix}-3\\-5\\1\end{bmatrix}}{\begin{bmatrix}-3\\-5\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\-5\\1\end{bmatrix}}\begin{bmatrix}-3\\-5\\1\end{bmatrix} + \frac{\begin{bmatrix}5\\-9\\5\end{bmatrix}\cdot\begin{bmatrix}-3\\2\\1\end{bmatrix}}{\begin{bmatrix}-3\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\2\\1\end{bmatrix}}\begin{bmatrix}-3\\2\\1\end{bmatrix} \\
&= \frac{35}{35}\begin{bmatrix}-3\\-5\\1\end{bmatrix} + \frac{-28}{14}\begin{bmatrix}-3\\2\\1\end{bmatrix} = \begin{bmatrix}3\\-9\\-1\end{bmatrix}.
\end{aligned}$$

The distance from $\mathbf{y}$ to the plane is the length of

$$\begin{bmatrix}5\\-9\\5\end{bmatrix} - \begin{bmatrix}-3\\-9\\-1\end{bmatrix} = \begin{bmatrix}8\\0\\6\end{bmatrix}.$$

The length (which is our final answer) is then

$$\sqrt{8^2+0^2+6^2} = 10.$$

Problems for F 11/22:

6.4.1 *The given set is a basis for a subspace* W*. Use the Gram–Schmidt process to produce an orthogonal basis for* W*.*

$$\begin{bmatrix}3\\0\\-1\end{bmatrix}, \quad \begin{bmatrix}8\\5\\-6\end{bmatrix}.$$

We want

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \frac{\begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}.$$

6.4.9 *Find an orthogonal basis for the column space of the matrix*

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

These are all orthogonal, so looks like we got it right.

6.4.10 *Find an orthogonal basis for the column space of the matrix below.*

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

This is identical to the one above.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

6.4.13 *The columns of Q below were obtained by running Gram–Schmidt orthonormalization on the columns of A. Find an upper-triangular R with $A = QR$.*

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

The rule is $R = Q^T A$. This gives

$$R = Q^T A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}.$$