Math 310, Lesieutre
Problem set #2
September 9, 2015

Problems for M 8/31:

1.3.11 *Determine if* $\mathbf{b}$ *is a linear combination of* $\mathbf{a}_1$, $\mathbf{a}_2$, *and* $\mathbf{a}_3$, *where*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

We need to form the matrix

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

and put it into echelon form to see whether there are any solutions.

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If all we're interested in is whether there are solutions are not (and not specifically what they are), reaching echelon form is enough – we don't need to go all the way to reduced echelon form. In this case we have no rows like $[000|b]$, so there are solutions, and $\mathbf{b}$ is indeed a linear combination of the $\mathbf{a}_i$.

1.3.18 *Define*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}.$$

For what values of h *is* $\mathbf{y}$ *in the plane generated by* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*?*

To figure this out, we'll find the reduced echelon form of the matrix, but in terms of h. Then we can see which values of h make us end up with a row $[00|b]$.

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & -14 & -3+2h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}.$$

We've again reached echelon form, which is enough to check whether or not there are solutions. In this case, if $7+2h$ isn't 0, we have one of the infamous $[00|b]$ rows, which means there are no solutions, and $\mathbf{y}$ is not in the span. On the other hand, if $7+2h=0$, then the system is consistent.

This gives us our answer: if $h=-7/2$, then $\mathbf{y}$ is in the span. If $h \neq -7/2$, then $\mathbf{y}$ is not in the span.

1.4.1 *Compute the product $A\mathbf{x}$ using (a) the definition and (b) the row-vector method. If the product is undefined, say why.*

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

The matrix A has 2 columns, whereas the vector $\mathbf{x}$ has three entries. There is a size mismatch: we can't multiple these two things.

1.4.3 *Compute the product $A\mathbf{x}$ using (a) the definition and (b) the row-vector method. If the product is undefined, say why.*

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

This sizes here match, so this one we can do.

(a) First we compute it using the definition, in terms of linear combinations of the columns.

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = (2) \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} + (-3) \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

(b) Using the other method,

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} (2)(6)+(-3)(5) \\ (2)(-4)+(-3)(-3) \\ (2)(7)+(-3)(6) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

We get the same answer either way, whcih is a relief.

1.4.10 *Write the system first as a vector equation and then as a matrix equation.*

$$\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$$

As a vector equation, this is:

$$x_1 \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

In matrix form, the same system is:

$$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

The book doesn't ask us to solve it, so I won't.

Problems for W 9/2:

1.5.5 *Write the solution set of the given homogeneous system in parametric vector form.*

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

First we need to find the solution set, using row reduction.

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution, using a parameter s for the free variable x_3, is:

$$\begin{aligned} x_1 &= 5s \\ x_2 &= -2s \\ x_3 &= s \end{aligned}$$

In parametric vector form, this is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

1.5.7 *Describe all solutions of* $A\mathbf{x} = \mathbf{0}$ *in parametric vector form, where* A *is row equivalent to the given matrix.*

$$A = \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}.$$

We have to form the augmented matrix and put it in rref, but that's easy: just add -3 times row 2 to row 1 and we get:

$$\left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

The variables x_3 and x_4 are free; let's assign them parameters s and t respectively.

The general solution is

$$\begin{aligned} x_1 &= -9s + 8t \\ x_2 &= 4s - 5t \\ x_3 &= s \\ x_4 &= t. \end{aligned}$$

In parametric vector form, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

1.5.9 *Same as above, but with*

$$A = \begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}.$$

First, rref:

$$\begin{bmatrix} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 0 \\ -1 & 3 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables x_2 and x_3 are free. Parametrize them by s and t.

$$\begin{cases} x_1 & = 3s - 2t \\ x_2 & = s \\ x_3 & = t \end{cases}$$

In parametric vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

1.5.15 *Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also give a geometric description of the solution set and compare it to that in Exercise 1.5.5.*

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

First step, as usual, is to find the general solution using row reduction.

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we get

$$\begin{aligned} x_1 &= -2 + 5s \\ x_2 &= 1 - 2s \\ x_3 &= s, \end{aligned}$$

which in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

This is the same thing we got way back in 1.5.5, but translated by a particular solution $\mathbf{v}_p = (-2, 1, 0)$.

Problems for F 9/4:

1.6.1 *Suppose an economy has only two sectors, Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find the equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenses.*

The matrix corresponding to this economy (as in the example on page 51) is

$$\begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix}$$

Goods' expenses are $0.2p_G + 0.7p_S$: this to buy 20% of its own output, and 70% of Services' output. This should be equal to p_G, its total income, so $0.2p_G + 0.7p_S = p_G$. Likewise for Services we should have $0.8p_G + 0.3p_S = p_S$. This is

$$\left[\begin{array}{rr|r} 0.8 & -0.7 & 0 \\ -0.8 & 0.7 & 0 \end{array}\right].$$

To get rref, add the first row to the second, which makes it 0. Then p_S is free, and $p_G = \frac{7}{8}p_S$.

1.6.5 *Balance the unbalanced chemical equation:*

$$B_2S_3 + H_2O \to H_3BO_3 + H_2S.$$

Let's give variable names to the missing coefficients:

$$x_1\mathrm{B_2S_3} + x_2\mathrm{H_2O} \to x_3\mathrm{H_3BO_3} + x_4\mathrm{H_2S}.$$

Each of the elements B, S, H, O gives us an equation involving this variables. In this order, we have

$$\begin{aligned} 2x_1 &= x_3 \\ 3x_1 &= x_4 \\ 2x_2 &= 3x_3 + 2x_4 \\ x_2 &= 3x_3 \end{aligned}$$

Writing down the system and running row reduction (please forgive me for not writing out all the steps this time),

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variable x_4 is free, which gives a general solution

$$\begin{cases} x_1 = \frac{1}{3}x_4 \\ x_2 = 2x_4 \\ x_3 = \frac{2}{3}x_4 \\ x_4 \text{ is free.} \end{cases}$$

Let's plug in 3 for x_4 to make everything integers: $x_1 = 1$, $x_2 = 6$, $x_3 = 2$, $x_4 = 3$. So the balanced equation is

$$\mathrm{B_2S_3} + 6\,\mathrm{H_2O} \to 2\,\mathrm{H_3BO_3} + 3\,\mathrm{H_2S}.$$

1.6.6 *Balance the unbalanced chemical equation:*

$$Na_3PO_4 + Ba(NO_3)_2 \to Ba_3(PO_4)_2 + NaNO_3.$$

This works a lot like the previous problem. The elements this time are Na, P, O, and Ba. Give names to the unknowns:

$$x_1\mathrm{Na_3PO_4} + x_2\mathrm{Ba(NO_3)_2} \to x_3\mathrm{Ba_3(PO_4)_2} + x_4\mathrm{NaNO_3}.$$

Let's write the equations straight into the matrix:

$$\begin{array}{r} \text{Na:} \\ \text{P:} \\ \text{O:} \\ \text{Ba:} \end{array} \left[\begin{array}{cccc|c} 3 & 0 & 0 & -1 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

Rref for this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

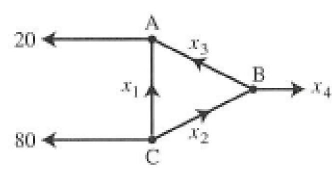
This means that the general solution, in terms of the free variable x_4 is

$$\begin{cases} x_1 = \frac{1}{3}x_4 \\ x_2 = \frac{1}{2}x_4 \\ x_3 = \frac{1}{6}x_4 \\ x_4 \text{ is free.} \end{cases}$$

Plugging in 6 for x_4, we obtain the balanced equation:

$$2\mathrm{Na_3PO_4} + 3\mathrm{Ba(NO_3)_2} \to \mathrm{Ba_3(PO_4)_2} + 6\mathrm{NaNO_3}.$$

1.6.11 *Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the largest possible value for x_3?*



Correction: the branch going out of C and labeled 80 *should instead go in to C.*

We get equations from each of the three nodes, plus one equation for the total:

$$\begin{aligned} A: &\quad x_1 + x_3 = 20 \\ B: &\quad x_2 = x_3 + x_4 \\ C: &\quad x_1 + x_2 = 80 \\ T: &\quad x_4 + 20 = 80 \end{aligned}$$

In matrix form, this is the equation:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 80 \\ 0 & 0 & 0 & 1 & 60 \end{pmatrix}.$$

Putting this in rref, we obtain

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The variable x_3 is free, and the general solution is

$$\begin{cases} x_1 = 20 - x_3 \\ x_2 = 60 + x_3 \\ x_3 \text{ is free} \\ x_4 = 60. \end{cases}$$

The maximum possible value of x_3 is 20: if x_3 were any larger than this, then x_1 would be negative.