Math 310, Lesieutre
Problem set #3
September 16, 2015

Problems for W 9/9:

1.7.1 *Determine if the following vectors are linearly independent:*

$$\begin{bmatrix}5\\0\\0\end{bmatrix}, \quad \begin{bmatrix}7\\2\\-6\end{bmatrix}, \quad \begin{bmatrix}9\\4\\-8\end{bmatrix}.$$

We're asking if there's a nonzero solution to

$$x_1\begin{bmatrix}5\\0\\0\end{bmatrix} + x_2\begin{bmatrix}7\\2\\-6\end{bmatrix} + x_3\begin{bmatrix}9\\4\\-8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This is a question for rref!

$$\left[\begin{array}{ccc|c}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & -6 & -8 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & 0 & 4 & 0\end{array}\right].$$

This is in echelon form, and we can already see that there are pivots in every column, hence no free variables. So there is a unique solution to the linear system, namely $\mathbf{x} = 0$, which means that the vectors are linearly independent.

1.7.4 *Determine if the following vectors are linearly independent:*

$$\begin{bmatrix}-1\\4\end{bmatrix}, \quad \begin{bmatrix}-2\\-8\end{bmatrix}.$$

Two vectors are linearly independent if and only if they are not parallel. These aren't parallel, so they are linearly independent.

1.7.9 *For what values of h is $\mathbf{v}_3$ in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$? Also, for what values of h is $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ linearly* dependent*? (Think about why this happens!)*

$$\begin{bmatrix}1\\-3\\2\end{bmatrix}, \quad \begin{bmatrix}-3\\9\\-6\end{bmatrix}, \quad \begin{bmatrix}5\\-7\\h\end{bmatrix}.$$

First let's figure out whether the given vector is in the span. We're asking if there's a solution to

$$x_1\begin{bmatrix}1\\-3\\2\end{bmatrix} + x_2\begin{bmatrix}-3\\9\\-6\end{bmatrix} = \begin{bmatrix}5\\-7\\h\end{bmatrix}.$$

This is again something that can be checked via row reduction.

$$\left[\begin{array}{rr|r} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 2 & -6 & h \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{array}\right].$$

This is echelon form, and there's a row [00|8]. That means that the system is inconsistent – it doesn't matter what h is. So there are no solutions, which means that the third vector is not in the span of the first two.

Whether the vectors are dependent is a slightly different question. We want to know if there is a solution to

$$x_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This can also be done by row reduction:

$$\left[\begin{array}{rrr|r} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{array}\right]$$

In this case, this isn't really necessary: just notice that $\mathbf{v}_2 = (-3)\mathbf{v}_1$, which means that $(3)\mathbf{v}_1 + (1)\mathbf{v}_2 + (0)\mathbf{v}_3 = 0$, which shows that the vectors are not independent.

1.7.29 *Find examples of* 3×2 *matrices* A *and* B *such that* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution, and such that* $B\mathbf{x} = \mathbf{0}$ *has a nontrivial solution.*

What does it mean for $A\mathbf{x} = \mathbf{0}$ to have anly the zero solution? It means that the columns of A are linearly independent. So if we pick A to have any two columns that aren't parallel, it will do the trick. For example,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

To make sure that $B\mathbf{x} = \mathbf{0}$ has other solutions, we should pick things so that the columns of B are not linearly independent. For example, they can be two parallel vectors:

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Problems for F 9/11:

1.8.1 *Let*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T(\mathbf{x}) = A\mathbf{x}$*. Find the images under* T *of*

$$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

We have

$$T(\mathbf{u}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

1.8.3 *Let* T *be defined by* $T(\mathbf{x}) = A\mathbf{x}$*, where*

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 7 \\ 3 & -2 & -5 \end{bmatrix}.$$

Find a vector $\mathbf{x}$ *whose image under* T *is* $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$*, and determine whether* $\mathbf{x}$ *is unique.*

The question is whether there is a solution to $A\mathbf{x} = \mathbf{b}$, and if so whether there is only 1 or infinitely many. Both of these are answered using rref.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ -2 & 1 & 7 & 7 \\ 3 & -2 & -5 & -3 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{13}{7} \\ 0 & 1 & 0 & \frac{5}{7} \\ 0 & 0 & 1 & \frac{10}{7} \end{array}\right].$$

There are no free variables, so there is a single solution

$$\mathbf{x} = \begin{bmatrix} \frac{13}{7} \\ \frac{5}{7} \\ \frac{10}{7} \end{bmatrix}.$$

1.8.15 *Define*

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ *and their images under* T*. Describe geometrically what* T *does to each vector* $\mathbf{x}$ *in* $\mathbb{R}^2$*.*

We have

$$T(\mathbf{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$T(\mathbf{v}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix},$$

The matrix projects a vector onto the y-axis.

1.8.16 *Same deal as the previous one, but with*

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This time,

$$T(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$
$$T(\mathbf{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix},$$

This map swaps the x and y coordinates. Geometrically, it is reflection across the line $y = x$.