Math 310, Lesieutre
Problem set #4
September 23, 2015

Problems for M 9/14:

1.9.2 *Suppose we have a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, with $T(\mathbf{e}_1) = (1,3)$, $T(\mathbf{e}_2) = (4,-7)$, and $T(\mathbf{e}_3) = (-5,4)$ (where $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$). Find the standard matrix for T. (Note: I originally typed the problem for 1.9.1 instead; if you already did that one, that's OK.)*

To do this, we just follow our recipe for writing down the matrix of a linear transformation. Since the map goes from $\mathbb{R}^3$ to $\mathbb{R}^2$, it should be given by a 2×3 matrix. The first column should be $T(\mathbf{e}_1)$, and they tell us what that vector is. The second is $T(\mathbf{e}_2)$, which we're also told. At last, we want $T(\mathbf{e}_3)$, which is given as well. So we stick it all together and get

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$$

1.9.4 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates points (about the origin) through $-\pi/4$ radians (i.e. $\pi/4$ radians, clockwise). Hint: $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2})$.*

To find the matrix, we need to figure out what are $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, and then we stick those things in as the columns.

What's $T(\mathbf{e}_1)$? Well, we are supposed to take the vector $(1,0)$ and rotate it $\pi/4$ radians clockwise (that's 45 degrees). Drawing a picture, you see that the result is $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2})$. You have to be a little careful here: you want a vector in the direction $(1,-1)$ (since that's what you get after rotating), but it should have the same length as $\mathbf{e}_1$, since rotation doesn't change length. That's where the $1/\sqrt{2}$ comes from.

Similarly we find that $T(\mathbf{e}_2)$ is $(1/\sqrt{2}, 1/\sqrt{2})$. Combining these into a matrix yields:

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

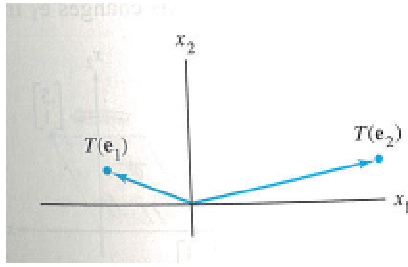
1.9.13 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2,1)$.*

Since the map is a linear transformation, we know that

$$T(2,1) = T(2\mathbf{e}_1 + \mathbf{e}_2) = 2\,T(\mathbf{e}_2) + T(\mathbf{e}_2).$$

So to find $T(2,1)$, draw $2T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ and find the sum by drawing a parallelogram.

A noteworthy feature fo this problem is that you can draw $T(2,1)$ without even knowing the precise coordinates of $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, using only the fact that the transformation is linear.



1.9.19 *Show that the transformation T below is a linear transformatino by finding a matrix that implements the mapping. Note that $x_1, x_2, \dots$ are not vectors but are entries in vectors.*

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3).$$

We want to find a matrix A that encodes this transformation. It has three input variables and two outputs, so it should be a 2×3 matrix, such that

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix}.$$

After writing the problem in this simpler way, you can see that the matrix we need is

$$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}.$$

Problems for W 9/16:

1.9.26 *Determine whether the linear transformation T from Exercise 1.9.2 is (a) one-to-one and (b) onto.*

That matrix was

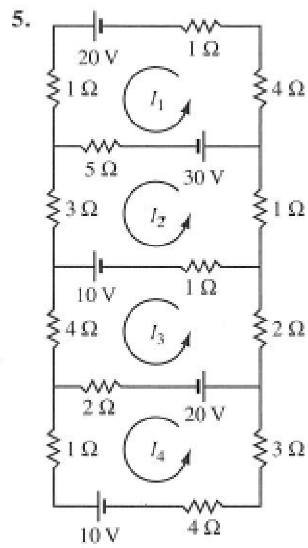
$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}.$$

Rref for the matrix is

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ 0 & -19 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) Since x_3 is a free variable when solving $A\mathbf{x} = \mathbf{0}$, the map is not one-to-one. (b) There is a pivot in every row, so the map is onto.

1.10.5 *Write a matrix that determines the loop currents in the depicted circuit. You don't need to solve for the loop currents.*



The formulas for each of the loops from top to bottom (assuming I get all my signs right) are

$$\begin{aligned} 11I_1 - 5I_2 &= 50 \\ -5I_2 + 10I_2 - I_3 &= -40 \\ -I_2 + 9I_3 - 2I_4 &= 30 \\ -2I_3 + 10I_4 &= -30 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 11 & -5 & 0 & 0 \\ -5 & 10 & -1 & 0 \\ 0 & -1 & 9 & -2 \\ 0 & 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 50 \\ -40 \\ 30 \\ -30 \end{bmatrix}$$

1.10.9 *In a certain region, about 7% of a city's population moves to the usrrounding suburbs each year, and about 5% of the suburban population moves into the city. In 2015, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where* $\mathbf{x}_0$ *is the initial population in 2015. Then estimate the populations in the city and in the suburbs two years later, in 2017.*

(See next week's solutions)

Problems for F 9/18:

2.1.1 *Consider the matrices*

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}.$$

Compute $-2A$, $B-2A$, AC, and CD.

This is fun. We have

$$
\begin{aligned}
-2A &= \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} \\
B-2A &= \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix} \\
AC &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \text{DIMENSION MISMATCH!!} \\
CD &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}
\end{aligned}
$$

2.1.3 *Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.*

$$
\begin{aligned}
3I_2 - A &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \\
(3I_2)A = 3(I_2A) = 3A &= \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}
\end{aligned}
$$

2.1.5 *Compute the product AB in two ways: (a) using the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately (b) using the row–column rule for computing AB. Here*

$$
A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}.
$$

By method (a), the first column of AB is

$$
(3)\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + (-2)\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ 12 \end{pmatrix}
$$

The second column is

$$
(-2)\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + (1)\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ -7 \end{pmatrix}
$$

So the product is

$$
AB = \begin{pmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{pmatrix}.
$$

This agrees with what we get by the row-column rule.

2.1.10 *Let* $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. *Verify that* $AB = AC$ *,even though* $B \neq C$.

We have

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$
$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

So $AB = AC$, even though $B \neq C$. Pretty weird.