Math 310, Lesieutre
Problem set #5
September 30, 2015

Problems for M 9/21:

2.2.1 *Find the inverse of*

$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}.$$

This matrix is 2×2, so we can just use the formula:

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}.$$

2.2.5 *Use the inverse from the first problem to solve the system*

$$8x_1 + 6x_2 = 2$$
$$5x_1 + 4x_2 = -1$$

We are trying to solve

$$A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

where A is the matrix from the first problem. The solution is

$$\mathbf{x} = A^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix},$$

so $x_1 = 7$, $x_2 = 9$.

2.2.31 *Find the inverse of the following matrix, using the row reduction method discussed in class:*

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

We write down the augmented matrix $[A|I_3]$ and run row reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right].$$

This means the inverse is

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

#4 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by rotating a vector 45 degrees clockwise. Last week, you found the matrix for T: call it A.*

(a) *Compute A^{-1}.*

The matrix we found was

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

The determinant is

$$ad - bc = \frac{1}{2} - \frac{-1}{2} = 1.$$

Then the formula for the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

(b) *Compute $A^{-1}\mathbf{v}$ for a couple vectors of your choice. How does A^{-1} transform vectors?*

I choose the vectors $(1, 0)$ and $(0, 1)$. Maybe you chose something else. This gives:

$$A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Plot these, and you can see what's going on: A^{-1} rotates a vector 45 degrees counterclockwise. That "undoes" A, which is what we expected.

Problems for W 9/23:

2.3.1 *Determine if the following matrix is invertible, using the method of your choice. Try to use as few calculations as possible:*

$$A = \begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$$

For 2×2, it's easy to check: look at $ad - bc$. Here it's $-30 - (-21) = -9$, which isn't 0, so the matrix is invertible.

2.3.3 *Same problem, with:*

$$A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

There are many ways to do this, including suffering through row reduction and seeing if you end up with the identity, which will never lead you astray but may waste your time. Here's my favorite alternative. Remember that A is invertible if A^T is; that's #12 on the list from lecture:

$$A^T = \begin{bmatrix} 5 & -3 & 8 \\ 0 & -7 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

So we need to figure out if A^T is invertible. But this matrix is in echelon form, and has 3 pivots. That means it's invertible, by #3 on the list! That's a lot easier than actually using row reduction to check.

2.3.13 *An* $m \times n$ upper triangular matrix *is one whose entries below the main diagonal are all* 0*. When is a square upper triangular matrix invertible? Justify your answer.*

An upper triangular matrix is automatically in echelon form. If all the entries on the diagonal are not 0, then there are n pivots, and that means the matrix is invertible.

On the other hand, if there is a 0 on the diagonal, then there are fewer than n pivots, which means that the matrix isn't invertible.

2.3.15 *Can a square matrix with two identical columns be invertible? Why or why not?*

Nope: we know that if a matrix A is invertible, its columns must be linearly independent. But if columns $\mathbf{a}_i$ and $\mathbf{a}_j$ are the same, then $(1) \cdot \mathbf{a}_i + (-1) \cdot \mathbf{a}_j = 0$, which shows that the columns are not linearly independent.

Problems for F 9/25:

1.10.9 *In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 5% of the suburban population moves into the city. In 2015, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where* $\mathbf{x}_0$ *is the initial population in 2015. Then estimate the populations in the city and in the suburbs two years later, in 2017. (This one is a repeat from last week, but now we actually covered it in lecture.)*

Translating the description of the situation into a formula, we're given that

$$\mathbf{x}_{n+1} = A\mathbf{x}_n$$

where

$$A = \begin{bmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 800000 \\ 500000 \end{bmatrix}$$

This gives

$$\mathbf{x}_1 = A = \begin{bmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{bmatrix} \begin{bmatrix} 800000 \\ 500000 \end{bmatrix} = \begin{bmatrix} 769000 \\ 531000 \end{bmatrix}$$

$$\mathbf{x}_2 = A = \begin{bmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{bmatrix} \begin{bmatrix} 769000 \\ 531000 \end{bmatrix} = \begin{bmatrix} 741720 \\ 558280 \end{bmatrix}$$

2.5.1 *Solve $A\mathbf{x} = \mathbf{b}$ using the LU decomposition:*

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}.$$

First we need to find the LU decomposition. The row reduction goes

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The steps we did were

(a) $R2- = (-1)R1$

(b) $R3- = 2R1$

(c) $R3- = (-5)R2$

The thing we ended with is U; the steps we did go into L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Now we're supposed to solve $A\mathbf{x} = \mathbf{b}$. To do that, we first solve $L\mathbf{y} = \mathbf{b}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 2 & -5 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & 16 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

So $\mathbf{y} = (-7, -2, 6)$. Now we solve $U\mathbf{x} = \mathbf{y}$.

$$\left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -7 & 0 & -19 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right].$$

So our solution is

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}.$$

You may think that we did a lot more work than we really had to do to solve a 3×3 system. You're probably right, but you really start to see the advantages when working with bigger systems of equations. Notice that all the row reductions we did were pretty easy, and involved adding 0 to things a lot of the time.

2.5.9 *Find an LU decomposition for the following matrix:*

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}.$$

Row reduction goes

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 9 & -5 & 6 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}.$$

The steps were

(a) $R2- = (-1)R1$

(b) $R3- = (3)R2$

(c) $R2- = (2/3)R3$

Plugging in the numbers from (a), (b), (c) to get L, and taking U as the outcome of reduction, the factorization is $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & \frac{2}{3} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}.$$

If you'll multiply, you'll find that this works (I hope).