Math 553, Lesieutre
Macaulay2 problems
April 27, 2015

1. (a) Consider the cubic $C \subset \mathbb{P}^2$ determined by $Y^2Z = X^3 - XZ^2$. Is the cubic smooth? Check your answer with Macaulay. Compute the genus of the curve.
   (b) Consider the cubic $C \subset \mathbb{P}^2$ determined by $Y^2Z = X^3 - X^2$. Is the cubic smooth? Check your answer with Macaulay. How would you get your hands on a resolution of C?
   (c) Let C be the intersection of the cubic $W^3 + X^3 + Y^3 + Z^3 = 0$ with a general quadric in $\mathbb{P}^3$. What is the genus of C?

2. In the RTG learning seminar, we learned how to compute the Hodge diamond of a quintic threefold using the Euler sequence and some characteristic class tricks. Let's see how to compute this in Macaulay2.
   (a) Construct the Fermat quintic as a variety over $\mathbb{Q}$.
   (b) Construct the sheaves Ω^i_X $(0 \leq i \leq 3)$ on X, and compute their cohomology.
   (c) This calculation was for a specific quintic hypersurface, over the field $\mathbb{Q}$. Would we have gotten the same answer over $\mathbb{C}$? What if we had used some other smooth quintic?

3. Let $C \subset \mathbb{P}^3$ be a rational normal curve. Find the normal bundle of C. (Hint: $C \cong \mathbb{P}^1$, so we know what all the possible rank-2 bundles are. It may not be easy to get Macaulay to fit this bundle into our classification – how can you determine which of the possibilities is the right one?)

4. Suppose that $p_1, \ldots, p_7$ are 7 general points in $\mathbb{P}^2$. What is the minimum degree of a homogeneous polynomial that vanishes to order at least 3 at each of the points?

5. The Grassmannian $G(2,4)$ is the variety parametrizing 2-dimensional subspaces of a 4-dimensional vector space V. It can be realized as a subspace of $\mathbb{P}\left(\bigwedge^2(V)\right) \cong \mathbb{P}^5$ via the map sending the subspace spanned by v and w to $v \wedge w$. Find the defining equations of the image.

6. Suppose that X is a variety over k. We say that X has *rational singularities* if $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$ whenever $f : Y \to X$ is a birational map from a smooth variety Y.
   (a) Consider the singular quadric $x^2 + y^2 = z^2$ in $\mathbb{P}^3$ (with coordinates $[w, x, y, z]$). Can you write down a smooth variety Y with a morphism to X?
   (b) Check that $R^1 f_* \mathcal{O}_Y = 0$ for your resolution. Hint: the vanishing condition is local on the base, so reduce to the case that X is affine and compute the derived functor as an H^1.
   (c) Consider the variety $X \subset \mathbb{P}^3$ determined by $x^3 + y^3 + z^3 = 0$, the cone over the Fermat cubic. Show that X does *not* have rational singularities.