Math 553, Lesieutre
Problem set #10
due April 3, 2016

1. If you didn't do this one last week, it's no longer optional:

   We know how to compute the canonical class of a hypersurface in $\mathbb{P}^n$ using the adjunction formula. This can also be done by writing down a form in local coordinates.

   Suppose that $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface defined by the equation $F(x_0, \ldots, x_{n+1}) = 0$. Let's consider the open set U_0 where $x_0 \neq 0$. The defining equation for X in U_0 is of the form $G(x_{10}, \ldots, x_{n+1,0}) = 0$, where $x_{i0} = x_i/x_0$.

   Let $V_i \subset U_0$ be the open set $U_0 \cap U_i$, i.e. the points for which the coordinate x_{i0} is nonzero.

   A basis for $\Gamma(V_i, \omega_X|_{V_i})$ is given by the single form

   $$\frac{(-1)^i}{\partial G/\partial x_{i0}} dx_{10} \wedge \cdots \wedge \widehat{dx_{i0}} \wedge \cdots dx_{0,n+1}.$$

   (a) Check that $\omega_i = \omega_j$ on the overlap $V_i \cap V_j$. Hence the ω_i piece together to a global n-form on the set $U_0 \subset X$

   (b) The form ω is regular on U_0, but it has other poles outside of this set. Compute the divisor of ω. What is the canonical class of X?

   (c) Compute the geometric genus $\dim \Gamma(X, \omega_X)$.

   (d) Use your answer to conclude that two hypersurfaces X, X' in $\mathbb{P}^{n+1}$ of degrees at least $n+1$ are not birational.

2. Suppose that X is a hypersurface of degree d in $\mathbb{P}^n$. What is the Kodaira dimension of X?

3. On the last homework, you showed that a complete intersection curve in $\mathbb{P}^3$ cut out by hypersurfaces of degrees d and e has $p_g(C) = \frac{1}{2}de(d+e-4)+1$. You also computed the canonical class of such a curve. Compute the degree of the canonical class in terms of the genus.

   (More work is needed to conclude that this formula works in general, since not every isomorphism class of curve can be realized as such a complete intersection.)

4. Find ample divisors on:

   (a) The blow-up of $\mathbb{P}^2$ at four general points.

   (b) The blow-up of $\mathbb{P}^2$ at three collinear points.

   Prove that the divisors are ample. You may use the Nakai criterion.

5. Here's a review question. Suppose that S is a graded ring.

(a) What are the affine schemes used to construct $\operatorname{Proj} S$?

(b) What are the points of $\operatorname{Proj} S$? (Both as ideals in S, and ideals in the rings defining the distinguished affines?)

(c) What are the sections of $\mathcal{O}(1)$ on the distinguished affines?

(d) Various theorems require that S be generated by S_1 as an S_0-algebra. What's the point of this assumption?

(e) The embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$ as a rational normal curve is given in coordinates by $[S, T] \mapsto [S^n, S^{n-1}T, \ldots, ST^{n-1}, T^n]$. How can you get this map using a map between graded rings?