Math 553, Lesieutre
Problem set #13
due April 24, 2016

1. Suppose that X is a ringed space, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, and $U \subset X$ is open. Show that $\mathscr{E}xt^i_X(\mathcal{F},\mathcal{G})|_U \cong \mathscr{E}xt^i_U(\mathcal{F}|_U,\mathcal{G}|_U)$.

2. Compute the dimension of $\operatorname{Ext}^i(\mathcal{O}_{\mathbb{P}^r}(a),\mathcal{O}_{\mathbb{P}^r}(b))$.

3. Every vector bundle on $\mathbb{P}^1$ is isomorphic to a sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(a_i)$. Some people call this the "Birkhoff–Grothendieck theorem", and others complain that this was known in some form long before Grothendieck was born. Let's prove it.

   (a) Suppose that E is a rank r vector bundle on $\mathbb{P}^1$. Let n be the largest number for which $H^0(\mathbb{P}^1,E(-n))$ is nonempty. Why is there such a n?

   (b) Let $d = \dim H^0(\mathbb{P}^1,E(-n))$. A section $s \in H^0(\mathbb{P}^1,E(-n))$ gives rise to a map $\mathcal{O}_{\mathbb{P}^1} \to E(-n)$. However, the cokernel of this map (as coherent sheaves) might *a priori* fail to be locally free. Explain why it is.

   (c) Hence we get a short exact sequence

   $$0 \longrightarrow \mathcal{O} \longrightarrow E(-n) \longrightarrow F \longrightarrow 0$$

   with F a vector bundle. What is $\dim H^0(\mathbb{P}^1,F)$? Show that if $d=1$, the short exact sequence must split.

   (d) Suppose instead that $d>1$. Then you can try to split F instead, since there are fewer sections. Conclude that $E \cong \mathcal{O}(n)^{\oplus d} \oplus G$ for some r and vector bundle G.

   (e) Conclude the result by induction on the rank.

4. (a) Let $L \subset \mathbb{P}^3$ be a line. Let's figure out its normal bundle. Pick a plane $S \subset \mathbb{P}^3$ containing L. Then we have an exact sequence

   $$0 \longrightarrow N_{L/S} \longrightarrow N_{L/\mathbb{P}^3} \longrightarrow N_{S/\mathbb{P}^3}|_L \longrightarrow 0$$

   Identify the line bundles the left and right (I don't mean the "0"s). What is $N_{L/\mathbb{P}^3}$?

   (b) Let L be the same line as above, but now let $Q \subset \mathbb{P}^3$ be a quadric containing L as a ruling. This time we get

   $$0 \longrightarrow N_{L/Q} \longrightarrow N_{L/\mathbb{P}^3} \longrightarrow N_{Q/\mathbb{P}^3}|_L \longrightarrow 0$$

   Identify all the sheaves appearing in this sequence, and write out the long exact sequence in cohomology. What are the dimensions of the spaces? Is this sequence split?

   (c) Identify the extension class $\delta(1)$ in $\operatorname{Ext}^1(N_{Q/\mathbb{P}^3}|_L, N_{L/Q})$ associated to the preceding normal bundle sequence.

5. How is your paper coming? Let me know if you'd like to set a time to meet.