Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #2: More dynamical degrees & examples
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Problem 1. a) Prove that if $\phi : X \dashrightarrow X$ is a birational transformation and $n = \dim(X)$, then $\lambda_{n-d}(\phi) = \lambda_d(\phi^{-1})$.

b) Let $f \colon X \dashrightarrow X$ and $g \colon Y \dashrightarrow Y$ be (bi)rational maps and let $\pi \colon X \dashrightarrow Y$ be a generically finite map such that $\pi \circ f = g \circ \pi$. Show that $\lambda_p(f) = \lambda_p(g)$ for all p.

c) Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold. Let D be a leading eigenvector of $\phi^* : N^1(X) \to N^1(X)$, and D' a leading eigenvector of $(\phi^{-1})^* : N^1(X) \to N^1(X)$. Show that either $D^2 = 0$ or $(D')^2 = 0$ (as elements of $N^2(X)$, or $H^{2,2}(X)$). (Hint: you can assume $\lambda_1(f)$ is a real eigenvalue)

Problem 2. Suppose that $D \subset \mathbb{P}^3$ is a surface of degree d, with multiplicities m_1, m_2, m_3, and m_4 at the four coordinate points. Compute the degree and multiplicities of $\mathrm{Cr}(D)$, where Cr is the standard Cremona involution. What if D is a curve instead of a surface?

Problem 3. a) Compute the dynamical degrees for the following affine maps: $(x, y) \mapsto (x^2y, xy)$, $(x, y) \mapsto (xy, y)$. What about the maps $(x, y) \mapsto (x^a y^b, x^c y^d)$ more generally?

b) Can you prove that these maps have positive entropy if the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has an eigenvalue greater than 1?

Problem 4. a) Describe two configurations of points in $\mathbb{P}^2$ which do not simply differ by a linear automorphism, but for which the corresponding blow-ups are isomorphic (hint: it might be easier to think of this in reverse – start with a rational surface, and describe two different ways to blow it down to $\mathbb{P}^2$).

b) Prove that a very general configuration of n points in $\mathbb{P}^2$ (over $\mathbb{C}$) is "Cremona-general", in the sense that an arbitrary sequence of standard involutions centered at three-tuples among the points is well-defined.

c) Suppose that $\mathbf{p}$ is a very general configuration of 10 points in $\mathbb{P}^2$. Show that there exist infinitely many other configurations $\mathbf{q}$ such that no two $\mathbf{p}$ and $\mathbf{q}$ are projectively equivalent, but such that $X_{\mathbf{p}} \cong X_{\mathbf{q}}$.

Problem 5. Let $\phi_M : E \times E \to E \times E$ be a linear automorphism of a torus. Determine the k-periodic points. How does the number of periodic points grow with k?

Problem 6. a) Let $f\colon X \to X$ be a zero entropy automorphism. Show that we can define polynomial analogues of the dynamical degrees

$$d_p(f) = \limsup_{n\to+\infty} \frac{\log\|(f^n)^*_p\|}{\log n} \in \mathbb{N}.$$

b) $(*)$ Does this work for birational transformations?

c) Let $\dim(X) = 3$; show that $d_1(f) \leq 4$.

d) $(*)$ Prove or give a counterexample: suppose that $\phi : X \dashrightarrow X$ is a dominant rational map. Then $\lambda_i(\phi)$ is an algebraic integer.