Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #1: Dynamical degrees & entropy
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Problem 1. a) *Let $X = \mathbb{P}^1$ and let $\phi : X \to X$ be the map $z \mapsto z^d$. Compute the entropy $h_{top}(\phi)$ in two different ways. Do your answers agree?*

i) *Directly from the topological definition;*

We first show that $h_{\text{top}}(\phi) \geq \log d$: the unit circle is preserved by ϕ, therefore $h_{\text{top}}(\phi) \geq h_{\text{top}}(\phi|_{S^1})$. Now, identifying S^1 with $\mathbb{R}/\mathbb{Z}$, the action of $\phi|_{S^1}$ is $[\alpha] \mapsto [d \cdot \alpha]$, where $[\alpha]$ is the class of α modulo $\mathbb{Z}$. Writing the real number α in base d

$$\alpha = c_1 d^{-1} + c_2 d^{-2} + \dots$$

this corresponds to erasing the first digit and rescaling:

$$[d \cdot \alpha] = [c_2 d^{-1} + c_3 d^{-2} + \dots].$$

Now it is easy to show that, for $\epsilon = d^{-M}$, the number of (N, ϵ)-separated orbits is exactly d^{M+N}, and that therefore

$$h_{\text{top}}(\phi) \geq h_{\text{top}}(\phi|_{S^1}) = \log d.$$

To show the converse inequality, remark first that all orbits starting outside the unit circle converge either to 0 or to ∞. One can bound the number of (N, ϵ)-separated orbits in the interior of the unit ball by noticing that, for $r, r' < 1$,

$$|re^{i\theta} - r'e^{i\theta'}| \leq C(|r - r'| + d(\theta, \theta'))$$

for some constant $C > 0$ (here d is the standard distance on $\mathbb{R}/2\pi\mathbb{Z}$). This means that, since for $|z| < 1$ we have $|\phi^N(z)| \to 0$ as $N \to +\infty$, for big N the number of (N, ϵ)-separated orbits for ϕ and for $\phi|_{S^1}$ are essentially the same. The same argument can be applied to bound the number of orbits starting outside the unit ball.

ii) *Using the Gromov-Yomdin theorem.*

Since $H^0(\mathbb{P}^1, \mathbb{C})$ is the additive group of constant complex-valued functions on $\mathbb{P}^1$, the action of ϕ on it is trivial. Therefore in order to apply Gromov-Yomdin result we just need to compute the action of ϕ on $H^2(X, \mathbb{C})$ (or equivalently on $Pic(\mathbb{P}^1)$). Since $H^2(X, \mathbb{C})$ is generated by the class of a point $[p]$ and $\phi^*([p]) = d[p]$, the action of ϕ is the multiplication by d; the result follows from Gromov-Yomdin theorem.

b) *Show that no orbit of ϕ is dense for the usual topology, but that there exist points whose orbit is dense in S^1, and therefore Zariski-dense in $\mathbb{P}^1$.*

All the orbits either are contained in S^1 or converge to 0 or ∞. We can eplicitly construct a point α of $S^1 \cong \mathbb{R}/\mathbb{Z}$ whose orbit is dense in S^1: in base d we write

$$\alpha = 0.c_1 c_2 c_3 \dots$$

in such a way that every finite sequence of digits appears in the sequence (c_i). Since ϕ acts on $\mathbb{R}/\mathbb{Z}$ by erasing the first digit and then rescaling, this condition ensures that the orbit of α is dense in S^1.

c) *Let E be an elliptic curve, and let M be an element of* $\mathrm{SL}_2(\mathbb{Z})$ *with* $\phi_M : E \times E \to E \times E$ *the induced automorphism. Repeat part (a) for this map. (You can try the example* $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$*).*

Let x, y be complex coordinates on the two copies of E in X. The matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on $X = E \times E$ as $\phi_M(x, y) = (ax + by, cx + dy)$.

1. Suppose for simplicity that M is diagonalizable and pick a basis of eigenvectors (v_1, v_2). Remark that the eigenvalues have to be real, so that we can suppose v_1 and v_2 to be real as well. By cutting the torus X into small cubes in the directions v_1, iv_1, v_2, iv_2 one can find precise bounds on the number of (N, ϵ)-separated points and obtian the same result as in Gromov-Yomdin (see next point).
2. The cohomology space $H^{1,0}(X)$ is generated by dx, dy, and the action of ϕ_M^* on this basis is given by the matrix M^T. Let λ, λ^{-1} be the eigenvalues of M; since $H^{1,1}(X) = H^{1,0}(X) \otimes H^{0,1}(X)$, the eigenvalues of $\phi_M^* \colon H^{1,1}(X) \to H^{1,1}(X)$ are $|\lambda|^2, |\lambda|^{-2}$ and 1 (with multiplicity 2). Therefore, by Gromov-Yomdin,

$$h_{\text{top}}(\phi_M) = 2 \log |\lambda|,$$

   where λ is the maximal modulus eigenvalue of M.

Problem 2. *Show that* $\mathbb{P}^1 \times \mathbb{P}^1$ *does not admit an automorphism of positive entropy. Can you prove this for any other classes of varieties?*

There are a couple ways you could approach this. For one, we know what the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ is, and it's possible to just check that all these maps have entropy equal to 0.

A slicker approach is this: think directly about the action of automorphisms on $N^1(X)$, which is a 2-dimensional vector space. Any automorphism pulls back effective classes to effective classes, and so preserves the effective cone $\overline{\mathrm{Eff}}(X)$. In particular, the generators of the two extremal rays on $\overline{\mathrm{Eff}}(X)$ are either both fixed, or both exchanged. Either way, the automorphism ϕ^2 fixes both generators of the effective cone, and so acts be the identity on $N^1(X)$. This means that it has entropy 0, according to the Gromov–Yomdin theorem.

The same argument works on any variety for which the effective cone is a rational polyhedral cone (i.e. generated by some finite set of generators). This holds for many varieties, for example any Mori Dream Space (and hence any Fano variety).

Another easy category of varieties with no positive entropy automorphisms are varieties with Picard rank 1. In this case $N^1(X)$ is one-dimensional, and since an automorphism must fix the generator, the map has entropy 0.

In dimension 2, the dynamical degrees detect the existence of invariant fibrations for a birational map, according to the following theorem of Diller and Favre.

Theorem 1. Let X be a projective surface and let $f \colon X \to X$ be an automorphism. Then one of the following holds:

1. f^* is of finite order; in this case some iterate of f is isotopic to the identity (i.e. some $f^{\circ k}$ lies in $\mathrm{Aut}^0(X)$),
2. some iterate of f^* is unipotent of infinite order; in this case $\lambda_1(f) = 1$ and f preserves a fibration $\pi\colon X \to C$ onto a curve.
3. f^* is semi-simple; in this case $\lambda_1(f)$ is a Salem number (i.e. an algebraic integer whose conjugates over $\mathbb{Q}$ are $1/\lambda_1(f)$ and some complex numbers of modulus 1) and f does not preserve any fibration.

We won't give a full proof, but the next few problems demonstrate some aspects of this fact.

Problem 3. a) *Suppose that $f : X \to X$ is an automorphism of a smooth projective surface, and there is a fibration to a curve $\pi : X \to C$ and an automorphism $g : C \to C$ with $\pi \circ f = g \circ \pi$. Show that $\lambda_1(f) = 0$.*

Suppose by contradiction that $\lambda = \lambda_1(f) > 1$; since f is an automorphism, this means that there exists a non-trivial class $D \in N^1_{\mathbb{R}}(X)$ such that $f^*D = \lambda D$. Let $F \in N^1(X)$ be the class of a fibre of π, so that $f^*F = F$ and $F.F = 0$. Then

$$D.D = f^*D.f^*D = \lambda^2 D.D, \qquad D.F = f^*D.f^*F = \lambda D.F$$

and therefore $D.D = D.F = 0$.
By Hodge's index theorem, the intersection product on $N^1_{\mathbb{R}}(X)$ has signature $(1, \rho(X) - 1)$; in particular, the maximal dimension of a subspace $V \subset N^1_{\mathbb{R}}(X)$ on which the intersection product is identically 0 is 1. Therefore we have $F.F \neq 0$, which contradicts the definition of F.

b) *Let X be the blow-up of $\mathbb{P}^2$ at the base locus of a pencil of cubics. Show that every automorphism of X must preserve the resulting elliptic fibration, and so has entropy 0.*

Let $p_1, \ldots, p_n$ be the base points of the pencil, and let $E_1, \ldots, E_n$ be the corresponding exceptional divisors of the blow-up $\pi\colon X \to \mathbb{P}^2$. Any automorphism of X preserves the canonical divisor $K_X = \pi^*\mathcal{O}(-3) + \sum_{i=1}^n E_i$. The strict transform of a cubic of the pencil has linear class $-K_X$ in $\mathrm{Pic}(X)$; therefore, the image of the strict transform of a cubic in the pencil is again a global section of $-K_X$. Since two strict transforms are disjoint, we have $K_X^2 = 0$; in particular, any global section of $-K_X$ must be the strict transform of an element of the pencil (otherwise the self-intersection of $-K_X$ would be positive). This shows that any automorphism of X preserves the elliptic fibration whose fibres are the strict transforms of the cubics in the pencil.

Problem 4. *Let X, f be as in Theorem 1 and let*

$$\mathcal{C} = \{D \in N^1_{\mathbb{R}}(X), D.D \geq 0\}$$

be the positive cone for the intersection product.

a) *Show that f^* preserves a line in $\mathcal{C}$.*

By Hodge's index theorem, the intersection product on $N^1_{\mathbb{R}}(X)$ has signature $(1, \rho(X) - 1)$, i.e. there exist a linear coordinates $x_1, \dots x_n$ such that the product is $q(x) = x_1^2 - (x_2^2 + \cdots + x_n^2)$. Therefore, the projectivization $\mathbb{P}\mathcal{C} \subset \mathbb{P}N^1_{\mathbb{R}}(X)$ is homeomorphic to an $(n-1)$-dimensional closed ball (to see this, one can cut with the affine hyperplane $\{x_1 = 1\}$).
The linear automorphism f^* preserves $\mathcal{C}$, and therefore induces a continuous homeomorphism of $\mathbb{P}\mathcal{C}$ onto itself. By Brouwer's fixed point theorem, any continuous map from the closed ball into itself admits a fixed point, which shows the claim.

b) *Show that, if f^* preserves a line in the interior of $\mathcal{C}$, then f^* has finite order.*

Let $\mathbb{R}D \subset \mathcal{C}$ be a fixed line in the interior of $\mathcal{C}$; then, since $D.D = f^*D.f^*D \neq 0$, we must have $f^*D = D$ ($f^*D = -D$ is impossible because the half cone of $\mathcal{C}$ containing the ample classes is f^*-invariant as well), and in particular we can suppose that D is an integer class. Therefore f^* preserves D and its orthogonal space $D^\perp$, which is a hyperplane defined over $\mathbb{Q}$ on which the intersection form is defined negative.
Now, any element of the orthogonal group preserving a lattice has finite order: to see this, take any base of the lattice $e_1, \dots e_{n-1}$; f^* must preserve the set of elements of the lattice with norm greater or equal than $\min\{e_1.e_1, \dots e_{n-1}.e_{n-1}\}$. This set being finite, some iterate of f^* acts as the identity on it, and therefore on the whole $D^\perp$. Since $f^*D = D$, such an iterate is the identity.

c) *Show that if f^* preserves a single line in $\partial\mathcal{C}$, then some iterate of f^* is unipotent of infinite order, and that then $||(f^n)^*||$ grows as cn^2.*

Before giving the solution we will prove a lemma that will be useful later.

Lemma 1. *Let $f\colon X \to X$ be an automorphism of a surface and let $f^*\colon N^1_{\mathbb{R}}(X) \to N^1_{\mathbb{R}}(X)$ be the induced linear automorphism. If λ is an eigenvalue of f^* with modulus $\neq 1$, then so is λ^{-1}; furthermore, λ and λ^{-1} both have algebraic multiplicity 1 and are the only eigenvalues with modulus $\neq 1$.*

The Lemma is essentially a corollary of the fact that isotropic subspaces of $N^1_{\mathbb{R}}(X)$ endowed with the intersection form have dimension at most 1. First remark that if λ is an eigenvalue with modulus $\neq 1$, then so is $\bar{\lambda}$; if we had $\lambda \notin \mathbb{R}$ and $v \in N^1_{\mathbb{C}}(X)$ is an eigenvector, then v and $\bar{v}$ are not collinear and by the above remark the values $v.v, v.\bar{v}, \bar{v}.\bar{v}$ cannot be simultaneously 0. But this leads to a contradiction with the formula $v.w = f^*v.f^*w$, so that λ has to be real.
Now let λ, μ be two (real) eigenvalues with modulus $\neq 1$ and let v, w be two eigenvectors. Again $v.v, v.w, w.w$ cannot be simultaneously 0, but $v.v = \lambda^2(v.v) = 0$ and $w.w = \mu^2(w.w) = 0$, so that $v.w = \lambda\mu(v.w) \neq 0$ and $\lambda\mu = 1$. A similar proof also shows that the multiplicity of λ and λ^{-1} is at most 1.

Now, suppose that there is no preserved line in the interior of $\mathcal{C}$ and exactly one preserved line on the boundary of $\mathcal{C}$. By the lemma, if at least one of the eigenvalues of f^* had modulus $\neq 1$, then we would have two such eigenvalues and they would be real, so we would have two or more preserved lines on the boundary of $\mathcal{C}$. Therefore all eigenvalues have modulus 1.
Since f^* preserves the lattice $N^1(X)$, the eigenvalues are algebraic integers of modulus 1 all of whose conjugates also have modulus 1. By Kronecker's lemma, they all have to be roots of unity, so that all the eigenvalues of some iterate of f^* are equal to 1 (which is the same as saying that such an iterate is unipotent).
Now, suppose by contradiction that the growth of $||(f^n)^*||$ is at least cn^3, i.e. the maximal Jordan block of f^* has dimension at least 4; this means that the restriction of f^* (or some

iterate) to some subspace of $N^1_{\mathbb{R}}(X)$ of dimension 4 is represented by the following matrix with respect to a basis v_1, v_2, v_3, v_4:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$J^n \sim \begin{pmatrix} 1 & n & n^2/2 & n^3/6 \\ 0 & 1 & n & n^2/2 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now

$$v_4.v_4 = J^n v_4.J^n v_4 \sim \frac{1}{36}n^6(v_1.v_1) + \frac{1}{6}n^5(v_1.v_2) + \left(\frac{1}{4}(v_2.v_2) + \frac{1}{3}(v_1.v_3)\right) + \ldots,$$

so we must have

$$v_1.v_1 = v_1.v_2 = \frac{1}{4}(v_2.v_2) + \frac{1}{3}(v_1.v_3) = 0.$$

The same computation for $v_3.v_3$ leads to

$$(v_2.v_2) + \frac{1}{2}(v_1.v_3) = 0,$$

so that the restriction of the intersection form to the subspace generated by $v_, v_2$ is identically 0, a contradiction.

Now we only have to prove that the growth of $||(f^n)^*||$ cannot be linear. Now, f^* preserves the half cone $\mathcal{C}_+$ containing the ample classes. If by contradiction the growth is linear, pick any class v in the interior of the positive half-cone; there exists a class v_1 on the boundary of $\mathcal{C}$ such that $(f^n)^*v = v + nv_1$ for $n \in \mathbb{N}$, so that $v_1 \in \mathcal{C}_+$; but we also have $(f^{-n})^*v = v - nv_1$ for $n \in \mathbb{N}$, so that $v_1 \in \mathcal{C}_+$ as well, a contradiction.

NB: a similar proof shows that, in any dimension, if the growth of $||(f^n)^*||$ is polynomial, then the degree of polynomial growth is even.

d) *Show that if f^* preserves at least two lines in $\partial\mathcal{C}$ and no line in the interior of $\mathcal{C}$, then f^* is semi-simple and $||(f^n)^*||$ grows as $c\lambda^n$, where $\lambda = \lambda_1(f)$ is a Salem number. Furthermore, f^* preserves exactly two lines in $\mathcal{C}$, which are not defined over $\mathbb{Q}$.*

If two preserved lines had eigenvalue 1, then f^* would act as the identity on the plane generated by these lines, and in particular f^* would preserve a line in the interior of $\mathcal{C}$, a contradiction. Therefore, by the Lemma above, f^* has exactly two eigenvalues with modulus $\neq 1$ (which are real): λ and λ^{-1}. Since they are algebraic integers, they cannot be rational and therefore their eigenspaces are not defined over $\mathbb{Q}$. Finally, suppose there is a third preserved line $\mathbb{R}v$ on the boundary of $\mathcal{C}$ (with eigenvalue 1 by the Lemma) and let $\mathbb{R}v_+$ be the eigenspace for the eigenvector λ. By Hodge's index theorem we have $v.v_1 \neq 0$, and we have $v.v_+ = f^*v.f^*v_+ = \lambda(v.v_+)$, which leads to a contradiction.

Problem 5. *Let's show that if $\phi : X \to X$ is an automorphism with unbounded degree, then it preserves an elliptic fibration. You might want to assume X is a K3 surface the first time through.*

The reference here is again Diller–Favre [DF01].

a) *Show that there exists an integral nef class D with $\phi^* D = D$, $D^2 = 0$, $D \cdot K_X = 0$.*

Consider the limit of the sequence of divisors

$$D = \lim_{n\to\infty} \frac{\phi^{n*}(H)}{\|\phi^{n*}(H)\|},$$

where H is a fixed ample and we have chosen a norm $\|\cdot\|$ on $N^1(X)$. The limit exists since after normalizing this is a sequence of divisors in a compact subset of $N^1(X)$. The limit is surely nef, since each $\phi^{n*}(H)$ is nef. Since we are assuming that $|\lambda_1(\phi)| = 1$, it must be that $\phi^* D = D$. The question is, why is this an integral class, rather than just an $\mathbb{R}$-divisor?

To see this, think about the Jordan decomposition of ϕ^*. The set of divisors E (maybe not ample) for which the limit in question is D is an open set in $N^1(X)$, by simple linear algebra. So the vector is nothing more than the leading eigenvector of the largest Jordan block of the matrix for ϕ^*, which is certainly integral.

Note that

$$0 = \lim_{n\to\infty} \frac{1}{\|\phi^{n*} H\|}(H \cdot K_X) = \lim_{n\to\infty} \frac{\phi^{n*}(H)}{\|\phi^{n*}(H)\|} \cdot K_X = .D \cdot K_X.$$

b) *Show that $h^0(nD) \geq 2$ for sufficiently large n.*

In the case of a K3, this is immediate from Riemann-Roch:

$$h^0(D) + h^0(K_X - D) = \chi(\mathcal{O}_X) + h^1(D) = 2 + h^1(D).$$

But on a K3 it is impossible that $h^0(K_X - D) \geq 1$, since this is $h^0(-D)$ and D is a nonzero nef class. Hence we obtain $h^0(D) \geq 2$.

The case of a rational surface requires a harder look. I can't quite follow the argument of [DF01] in this case; you may find it helpful to consult [Gri16].

c) *Show that the rational map determined by nD is an elliptic fibration.*

Passing to a multiple, we might as well assume that $h^0(D) \geq 2$, and we get a map $\phi_D : X \dashrightarrow \mathbb{P}^1$. After removing the fixed component of $|D|$, we may assume the map ϕ_D is a morphism. Taking the Stein factorization, we get a map $\pi : X \to \mathbb{P}^1$ whose general fibers are irreducible. The genus of these fibers is 1 by adjunction, and so we have an elliptic fibration which is ϕ-invariant.

Problem 6. *Let E be an elliptic curve, and let M be an element of $\mathrm{SL}_2(\mathbb{Z})$ with $\phi_M : E \times E \to E \times E$ the induced automorphism. Describe the induced linear automorphism f^* and check the results of Theorem 1. In the semi-simple case, show that ϕ_M preserves a pair of smooth foliations $\mathcal{F}_+, \mathcal{F}_-$ whose leaves are dense in $E \times E$.*

If M has finite order, then we are in situation (a) of the Theorem.

If M (or some iterate) is unipotent, then we are in situation (b) of the Theorem. In some base of $\mathbb{C}^2$ we can write

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and then ϕ_M preserves the elliptic fibration whose fibres are given by $\{y = constant\}$.

Finally, if M is semi-simple, we have seen that $\lambda_1(\phi_M)$ is a Salem number (actually quadratic in this case). The foliations $\mathcal{F}_+, \mathcal{F}_-$ are given by the constant directions of the eigenspaces of M. The action on leaves of $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) is a homothety with factor λ (resp. λ^{-1}), which easily implies that $\mathcal{F}_+, \mathcal{F}_-$ are the only invariant foliations; since their leaves have irrational slope, they aren't the fibres of an invariant fibration, so that there are no invariant fibrations.

Problem 7. *Fix a smooth plane cubic $E \subset \mathbb{P}^2$, and let p be a general point on E. We may define a rational map $\tau_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as follows. Given a point $x \in \mathbb{P}^2$, draw the line L from x to p. Generically, this line meets E in two points. There is a unique involution of L that fixes these two points (why?). Let $\tau_p(x)$ be the image of x under this involution.*

The reference for this example is [Bla13], which also considers a higher-dimensional analog of this map.

a) *The map τ_p is a rational map, not an automorphism. Determine the indeterminacy locus of τ_p. Show that after blowing up the indeterminacy locus, we obtain a map $\tilde{\tau}_p : X_p \to X_p$ which is an involutive automorphism of a rational surface (or just trust us and go to (b)).*

Let's try to reason out where the map is not defined; for a proof using a local calculation, check out [Bla13]. The one thing that can go wrong in our geometric description is that the line L might meet E at only a single point, rather than two distinct points. In this case, the involution τ_p isn't defined. There are four lines L_i through p which are tangent to E, namely the translates of $-p$ by the half-periods of E. Let $p_1,\ldots,p_4$ be the four points where this tangency occurs.

Consider what happens on a line "very close" to L (in the complex topology, say). You can see that the line L_i must be contracted to the point p_i. The other way, the point p_i is blown up to the line L_i. After blowing up these four points, together with the point p itself, we obtain a morphism. (This is not a complete argument! See [Bla13]. But it's enough geometry for us to do the rest of the problem.)

b) *Compute the action of $\tilde{\tau}_p : X_p \to X_p$ on $N^1(X_p)$.*

The space $N^1(X_p)$ is 6-dimensional, generated by the classes H, E_0^p, E_1^p, E_2^p, E_3^p, and E_4^p. Unfortunately, it's not so clear what the images of *any* of these basis vectors are. On the other hand, if we can find the images of six spanning classes in $N^1(X_p)$, that's enough to find the matrix.

There are a few easy ones. We've seen geometrically that $E_i^p \leftrightarrow H - E_0^p - E_i^p$ (for $1 \leq i \leq 4$). We also know that the elliptic curve E is itself fixed (as it must be – it represents the anticanonical class of $X_\mathbf{p}$!) That's five classes: we still need one more. But a general line through the point p is also preserved, as is clear from the construction. This gives the six requisite classes, and it's now a straightforward matter of linear algebra to see that the matrix for the map is given with respect to the above basis by the map

$$\tau_p^* = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

It is probably healthy to do some sanity checks at this point and make sure we haven't gone totally off the rails. I invite you to check that $(\tau_p^*)^2 = 1$ and $\tau_p^* K_X = K_X$, as expected.
This shows that the image of a general line is a cubic with a node at p, which isn't so clear from our original description of the map. We also see that the image of the exceptional divisor E_0^p is the quadric passing through the 5 points implicated in this mess, which is at least plausible since it's a (-1)-curve.

c) *Show that if $q \in E$ is another point, there exists a rational surface X_{pq} such that τ_p and τ_q both lift to automorphisms of X_{pq}. Compute the matrices for the action of these involutions on $N^1(X_{pq})$.*
The map τ_p^* lifts to the blow-up, and it simply preserves each of the classes E_i^q. That means it's given by the matrix

$$\tau_p^* = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix for τ_q^* is likewise given by

$$\tau_q^* = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

d) *Check that $\tau_p \circ \tau_q$ acts with infinite order on $N^1(X_{pq})$, but this map has entropy* 0.
The product matrix is

$$\tau_p^*\tau_q^* = \begin{pmatrix} 9 & 2 & 1 & 1 & 1 & 1 & 6 & 3 & 3 & 3 & 3 \\ -6 & -1 & -1 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -2 \\ -3 & -1 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Does it have infinite order? With some computer assistance, we find that the Jordan decomposition is $\tau_p^*\tau_q^* = J\Lambda J^{-1}$, where

$$J = \begin{pmatrix} 4 & 6 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & -2 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The 3×3 Jordan block you see in the top left corner of Λ guarantees that the map has infinite order. However, there is no eigenvalue with norm larger than 1, so the entropy of the map is 0.

e) *This means that $\tau_p \circ \tau_q$ preserves an elliptic fibration: can you find it?*
The proof of the Diller–Favre/Gizatullin theorem above tells us what divisor must determine the elliptic fibration: it's the map given by the leading eigenvector of the 3×3 Jordan block. We

can read that off from the computation above, and conclude that the invariant elliptic fibration must be given by the linear system associated to

$$D = 4H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p - 2E_0^q - E_1^q - E_2^q - E_3^q - E_4^q.$$

This is the linear system of quartics in $\mathbb{P}^2$ with nodes and p and q, and passing through the other 8 blown up points.

It's easy to check that this numerical class is indeed preserved by $\tau_p^* \tau_q^*$. What's not so clear is that there is actually a pencil of such curves, or how to construct them (short of reading through the entire argument of Gizatullin). Here's one direct approach.

Let L denote the strict transform of the line joining p and q, and let E denote the elliptic curve. Observe that

$$\begin{aligned} &L \sim H - E_0^p - E_0^q \\ &E \sim 3H - E_0^p - E_1^p - E_2^p - E_3^p - E_4^p - E_0^q - E_1^q - E_2^q - E_3^q - E_4^q = -K_X \\ &F \sim L + E \end{aligned}$$

Now, there's a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_X(L+E) \longrightarrow \mathcal{O}_E(L+E) \longrightarrow 0$$

In cohomology, this gives

$$0 \longrightarrow H^0(X, \mathcal{O}_X(L)) \longrightarrow H^0(X, \mathcal{O}_X(L+E)) \longrightarrow H^0(E, \mathcal{O}_E(L+E))$$

$$\longrightarrow H^1(X, \mathcal{O}_X(L)) \longrightarrow H^1(X, \mathcal{O}_X(L+E)) \longrightarrow H^1(E, \mathcal{O}_E(L+E)) \longrightarrow$$

The first term is 1-dimensional, since L is an irreducible curve of self-intersection -1 (we blew up 2 points on a line). The second is what we're trying to figure out: remember that $L+E \sim D$, and we're trying to show that $H^0(X, \mathcal{O}_X(D)) > 1$. The last term in the first row is a bit more work. We have $E \cdot E = -1$ (we blew up ten points on a cubic) and $E \cdot L = 1$. So $(L+E)|_E$ is a degree 0 line bundle on E, and it either has a section or doesn't, depending on whether or not it's 0 in $\mathrm{Pic}^0(E)$.

However, $(L+E)|_E$ is less mysterious than $L+E$ itself: we noticed earlier that there these is a quadric Q_p in $\mathbb{P}^2$ which is tangent to E at p, and passes through the points $p_1, \ldots, p_4$. This quadric restricts to E as a section of $(2H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p)|_E$. Note that it doesn't give a section in $H^0(X, \mathcal{O}_X(3H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p))$: the quadric doesn't have a double point at p, but it is tangent to the curve there. The sum with the corresponding quadric for q gives a section of $4H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p - 2E_0^q - E_1^q - E_2^q - E_3^q - E_4^q$, i.e. of D.

Hence the dimensions across the first row are

$$0 \longrightarrow 1 \longrightarrow ??? \longrightarrow 1 \longrightarrow$$

If we can show that $H^1(X, \mathcal{O}_X(L)) = 0$, we're in business: this would give $\dim H^0(X, \mathcal{O}_X(L + E)) = 2$, as required. To check it, write down the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_L(L) \longrightarrow 0$$

In the long exact sequence, the group we want is sandwiched between $H^1(X, \mathcal{O}_X)$ (which is 0 since X is a rational surface), and $H^1(L, \mathcal{O}_L(L)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. Unwinding everything, we get $H^0(X, \mathcal{O}_X(D)) = 2$, as required. One section of the line bundle is the reduced curve given as the union of E and L, which is the image of the first map in our long exact sequence in cohomology. (I'm not sure how to see the second one geometrically.)

f) *Fix a third point r on E. Using the same construction, exhibit a positive entropy automorphism of a rational surface X_{pqr}.*

Now we get a blow-up of $\mathbb{P}^2$ at 15 points on E, and three involutions τ_p, τ_q, τ_r. I'm not going to write down all the matrices in this case, but the end result is that

$$\tau_p^*\tau_q^*\tau_r^* = \begin{pmatrix}
27 & 2 & 1 & 1 & 1 & 1 & 6 & 3 & 3 & 3 & 3 & 18 & 9 & 9 & 9 & 9 \\
-18 & -1 & -1 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -2 & -12 & -6 & -6 & -6 & -6 \\
-9 & -1 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-9 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-9 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-9 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-6 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -2 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & -1 & -1 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -1 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1
\end{pmatrix}$$

The characteristic polynomial is $(t-1)^4(t+1)^{10}(t^2 - 18t + 1)$, and the largest root is $t = 9 + 4\sqrt{5} \approx 17.94$. The entropy is then $\log(9 + 4\sqrt{5})$, which is positive. Phew.

Problem 8. a) *Construct a compact metric space X and a map $\phi : X \to X$ with $h_{top}(\phi) = \infty$. Can you find such a map when $X = [0, 1]$?*

We know the map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^d$ has entropy $\log d$. Now let $X = \prod_{i=1}^{\infty} \mathbb{P}^1$, which is metrizable, and compact by Tychonoff's theorem. Consider the map $\psi : X \to X$ given by $z \mapsto z^i$ on the ith factor. One can now construct (n, ϵ)-separated sets of points by placing all the points in the ith factor, which gives a lower bound of $\log i$ on the entropy for any i. It follows that the entropy is infinite.

It is also possible to find such a map when $X = [0, 1]$. One approach is to start with a sequence of functions $f_n : [0, 1] \to [0, 1]$ with $f_n(0) = 0$ and $f_n(1) = 1$ with $h_\top(f_n) \to \infty$. One can than use these to cook up a map of infinite entropy: define your function on $[0, 1/2]$ using f_1, then $[1/2, 3/4]$ using $f_2, \ldots$. There's a picture in [Mil, Figure 39], which is a nice source on the setup and basic properties of topological entropy.

b) *Show (from the definition) that if $\phi : X \to X$ is an automorphism of a variety, $h_{top}(\phi)$ is finite. (Hint: in fact, show that if $\phi : X \to X$ is Lipschitz with constant C on a manifold X,*

then $h_{top}(\phi) \leq C \dim X$*.)*
You can find a proof of this one in Katok & Hasseblatt, page 120 [KH95].

References

[Bla13] Jérémy Blanc, *Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces*, Indiana Univ. Math. J. **62** (2013), no. 4, 1143–1164.

[DF01] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), no. 6, 1135–1169.

[Gri16] Julien Grivaux, *Parabolic automorphisms of projective surfaces (after M. H. Gizatullin)*, Mosc. Math. J. **16** (2016), no. 2, 275–298.

[KH95] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza.

[Mil] John Milnor, *Introductory dynamics lectures. 7. dimension and entropy*, http://www.math.stonybrook.edu/ jack/DYNOTES/.

Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #2: More dynamical degrees & examples
John Lesieutre and Federico Lo Bianco
July 19, 2016

Problem 1. a) *Prove that if $\phi : X \dashrightarrow X$ is a birational transformation and $n = \dim(X)$, then $\lambda_{n-d}(\phi) = \lambda_d(\phi^{-1})$.*
If ϕ is an automorphism, then

$$(\phi^N)^* H^{n-d}.H^d = H^{n-d}.(\phi^N)_* H^d = H^{n-d}.(\phi^{-N})^* H^d,$$

and we obtain the result by passing to the limit.
If ϕ is only birational, we can find dense open subsets U_N, V_N such that ϕ^N induces an isomorphism $U_N \cong V_N$; the same computations as above work provided we restrict to U_N, V_N, and give exactly the same result (for a general theory on the pull-back of cohomology classes by rational maps, see [Dem97]).

b) *Let $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$ be (bi)rational maps and let $\pi\colon X \dashrightarrow Y$ be a generically finite map such that $\pi \circ f = g \circ \pi$. Show that $\lambda_p(f) = \lambda_p(g)$ for all p.*
Suppose first that π is an unramified covering of degree d and that f, g are biregular; if H_Y is an ample divisor, then so is $H_X := \pi^* H_X$, and we have

$$(f^N)^* H_X^p.H_X^{n-p} = \pi^*(g^N)^* H_Y^p.\pi^* H_Y^{n-p} = d \cdot \left((g^N)^* H_Y^p.H_Y^{n-p}\right),$$

so that the result follows by passing to the limit. In the general setting, one has to restrict to the dense open sets where the hypothesis we assumed stay true, and carry out the computations on those sets.

c) *Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold. Let D be a leading eigenvector of $\phi^* : N^1(X) \to N^1(X)$, and D' a leading eigenvector of $(\phi^{-1})^* : N^1(X) \to N^1(X)$. Show that either $D^2 = 0$ or $(D')^2 = 0$ (as elements of $N^2(X)$, or $H^{2,2}(X)$). (Hint: you can assume $\lambda_1(f)$ is a real eigenvalue)*
We have $\phi^* D = \lambda_1(f) D$ and $(f^{-1})^* D' = \lambda_1(f^{-1}) D' = \lambda_2(f) D'$ by point (a). If by contradiction we had $D^2 \neq 0 \neq (D')^2$, then since

$$f^* D^2 = \lambda_1(f)^2 D^2, \qquad (f^{-1})^*(D')^2 = \lambda_1(f^{-1})^2 (D')^2,$$

we would have $\lambda_2(f) \geq \lambda_1(f)^2$ and $\lambda_2(f^{-1}) = \lambda_1(f) \geq \lambda_1(f^{-1})^2 = \lambda_2(f)^2$, contradiction since $\lambda_1(f) > 1$.

Problem 2. *Suppose that $D \subset \mathbb{P}^3$ is a surface of degree d, with multiplicities m_1, m_2, m_3, and m_4 at the four coordinate points. Compute the degree and multiplicities of $\mathrm{Cr}(D)$, where Cr is the standard Cremona involution. What if D is a curve instead of a surface?*

Let me do this a couple different ways. The first is to attack it directly using the formula for the Cremona involution, which is $[W, X, Y, Z] \mapsto [W^{-1}, X^{-1}, Y^{-1}, Z^{-1}]$. Suppose our surface has degree d and is defined by an equation of the form

$$\sum_{i+j+k+l=d} a_{ijkl} W^i X^j Y^k Z^l$$

What's the multiplicity at each of the four standard coordinate points? To find the multiplicity at $[1, 0, 0, 0]$, you can pass to an affine chart where it's the origin by plugging in $W = 1$ to the formula. The multiplicity is then given by the lowest degree of any of the remaining terms in the sum. This term comes from the term in the original formula with the *largest* power of W, and we see that $m_1 = d - \mu_W$, where μ_W is the maximum power of W in any term. Similar formulas hold for the other multiplicities.

Now, the strict transform is defined by the formula

$$\sum_{i+j+k+l=d} a_{ijkl} \frac{1}{W^i X^j Y^k Z^l},$$

which doesn't really make sense until we clear denominators. To do this, we need to multiply through by $W^{\mu_W} X^{\mu_X} Y^{\mu_Y} Z^{\mu_Z}$. Observe that the resulting polynomial has degree

$$\begin{aligned} d' &= \mu_W + \mu_X + \mu_Y + \mu_Z - d \\ &= (d - m_1) + (d - m_2) + (d - m_3) + (d - m_4) - d = 3d - \sum_{i=1}^{4} m_i. \end{aligned}$$

What's the new multiplicity? Well, we can assume some term of the defining equation for our surface is not divisible by W. This means that the largest W factor in the equation for the strict transform is given by μ_W, and the new multiplicity is

$$m'_j = d' - \mu'_W = (3d - \sum_{i=1}^{4} m_i) - (d - m_j) = 2d - \sum_{i \neq j} m_i.$$

There is another, more geometric way to do the problem as well. What we're really trying to do is find the matrix for the map $N^1(X) \to N^1(X)$, where X is the blow-up of $\mathbb{P}^3$ at the four coordinate points. The map $X \dashrightarrow X$ is a pseudoautomorphism, and it suffices to find the images for a basis of $N^1(X)$.

There are four classes whose images we know: the exceptional divisor E_i is mapped to the strict transform of the plane through the points other than p_i, which has class $H - \sum_j E_j + E_i$. To get one more class, observe that the canonical class of X must be preserved by the pullback. This is enough information to find that the in the basis H, E_1, E_2, E_3, E_4, the matrix for the pullback is

$$\phi^* = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 0 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{pmatrix},$$

Reading across the rows, we recover the formulas we derived earlier.

Problem 3. a) *Compute the dynamical degrees for the following affine maps:* $(x, y) \mapsto (x^2y, xy)$, $(x, y) \mapsto (xy, y)$. *What about the maps* $(x, y) \mapsto (x^a y^b, x^c y^d)$ *more generally?*

The action of the space of matrices with integer coefficients on $\mathbb{P}^2$ by rational morphism given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x, y) = (x^a y^b, x^c y^d)$$

respects the matrix product: if A and B are matrices with integer coefficients, then $A(B(x, y)) = (AB)(x, y)$. Therefore, if we denote $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the action by a matrix A, f^n is the action by the matrix A^n. Therefore the growth of $||(f^n)^*||$ is given exactly by the growth of $||A^n||$ (for example with the maximal coefficient norm), which is $c\lambda^n$ where λ is the eigenvector with maximum modulus (or polynomial if some power of A is unipotent). Thus $\lambda_1(f) = \lambda$ where λ is the eigenvalue of A with maximum modulus.

b) *Can you prove that these maps have positive entropy if the matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *has an eigenvalue greater than* 1*?*

Note that, since f is not regular, Gromov-Yomdin's theorem cannot be directly applied. However, we can restrict f to $\mathbb{T} = S^1 \times S^1 \subset \mathbb{C}^2$ ($S^1 \subset \mathbb{C}$ being the unit circle); $\mathbb{T}$ is a topological torus, and if we pass to linear coordinates f acts as $(x_1, x_2) \mapsto (ax_1 + bx_2, cx_3 + dx_4)$; we can check explicitly that this map has positive entropy (see exercise 1.c of lecture 1 for a trace), thus f itself has positive entropy.

Problem 4. a) *Describe two configurations of points in* $\mathbb{P}^2$ *which do not simply differ by a linear automorphism, but for which the corresponding blow-ups are isomorphic (hint: it might be easier to think of this in reverse – start with a rational surface, and describe two different ways to blow it down to* $\mathbb{P}^2$*).*

Let $\mathbf{p}$ be a configuration of six points p_1, p_2, p_3, p_4, p_5, and p_6 with p_1, p_2, and p_3 collinear and the other three points chosen general. Let X_p be the blow-up. It is not too hard to check that there is a unique (-2)-curve on X_p, given by the strict transform of the line through the first three points.

The idea is that we can blow down X_p in another way. Contract the strict transforms of the three lines between p_4, p_5, and p_6, together with the exceptional divisors over the points p_1, p_2, and p_3. This yields a new configuration $\mathbf{q}$ of six points. What's the image of the (-2)-curve in $\mathbb{P}^2$? It's now a conic passing through all 6 points, by a computation similar to the one from lecture.

It follows that the new configuration can not be projectively equivalent to the old one: for $\mathbf{p}$ there is no conic through all of the points, while for $\mathbf{q}$ there is one. On the other hand, for $\mathbf{p}$ there are three points on a line, which is not the case for $\mathbf{q}$.

b) *Prove that a very general configuration of n points in* $\mathbb{P}^2$ *(over* $\mathbb{C}$*) is "Cremona-general", in the sense that an arbitrary sequence of standard involutions centered at three-tuples among the points is well-defined.*

Problems with Cremona transformations being defined only arise when three of our points become collinear after some sequence of transformations, which means that there is a (-2)-curve on the blow-up $X_{\mathbf{p}}$. Hence to show that a general configuration is Cremona-general, it's enough to show that for very general $\mathbf{p}$, there are no (-2)-curves. It is important here that we're working over $\mathbb{C}$ (or at least some other uncountable field), which the problem should have specified.

The trick is to specialize the points: it's enough to prove this for a single configuration of points, and it then holds for a very general one. Let Γ be a smooth cubic in $\mathbb{P}^2$, and blow up n points on Γ. Observe that the strict transform of Γ on X is an anticanonical divisor.

If there is a (-2)-curve C on the blow-up, then it satisfies $\Gamma \cdot C = 0$, by adjunction. Writing $C \sim dH - \sum_{i=1}^n m_i E_i$, and noting that $E_i|_\Gamma$ is the point p_i in $\mathrm{Pic}^0(\Gamma)$, we obtain

$$dH|_\Gamma \sim \sum_{i=1}^n m_i p_i$$

in $\mathrm{Pic}^0(\Gamma)$. The set of configurations $(p_i)_{1\leq i \leq n}$ in E^n for which this holds with any particular choice of d and the m_i's is a proper Zariski closed subset of E^n. Since the set of all possible d and m_i is only countable, we conclude that there are no (-2)-curves on the blow-up as long as the p_i are very general (i.e. outside of some countable union of Zariski closed subsets of the configuration space. (That there actually exist such points outside of such a countable union requires that we work over an uncountable field.)

c) *Suppose that* $\mathbf{p}$ *is a very general configuration of* 10 *points in* $\mathbb{P}^2$*. Show that there exist infinitely many other configurations* $\mathbf{q}$ *such that no two* $\mathbf{p}$ *and* $\mathbf{q}$ *are projectively equivalent, but such that* $X_\mathbf{p} \cong X_\mathbf{q}$*.*

The argument here very closely follows [Les15, Lemma 3] (which is in dimension 3), so I will be a little brief. The idea is to prove that some special configuration $\mathbf{p}$ has infinite orbit; the result for a very general configuration then follows. Choose our initial configuration so that the blow-up X has three points on a line, but no other (-2)-curves; this is possible using the construction from the previous exercise. Let C be this (-2)-curve.

Now we use exactly the construction that appeared in the lecture. We repeatedly make a Cremona transformation at the last three points, and then reorder the points to move these three points to the beginning of the list. The growth of the degrees is as follows:

d	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	0	0	0	0	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0	0	0
4	2	2	2	1	1	1	0	0	0	0
8	4	4	4	2	2	2	1	1	1	0
14	7	7	6	4	4	4	2	2	2	1

Remember that the Cremona involution gives isomorphism $X_\mathbf{p} \to X_\mathbf{q}$, so the surfaces we obtain are all isomorphic (the different degrees we measure for the image of C reflect the fact that we're blowing down different (-1)-curves, and so working in a different basis for $N^1(X)$.

Here the nth row is the class of the strict transform of C after the first $n-1$ Cremona transformations. For each configuration, there is a unique effective (-2)-curve class, but these classes all have different degrees. We conclude that these configurations are not projectively equivalent and so the Cremona orbit contains infinitely many nonisomorphic configurations of points.

One upshot of this problem is that it's not easy to define a moduli space of (non-minimal) rational surfaces in any reasonable way.

Problem 5. *Let $\phi_M : E \times E \to E \times E$ be a linear automorphism of a torus. Determine the k-periodic points. How does the number of periodic points grow with k?*

Let $E \cong \mathbb{C}/\Lambda$; finding the k-periodic points is the same as solving the equation $M^k v = v$ modulo Λ. Suppose first that ϕ_M has positive entropy (which is the same as asking that 1 is not an eigenvalue of M^k for any k). For each vector $w \in \Lambda \times \Lambda$, the equation $M^k v = v + w$ has exactly one solution $v \in \mathbb{C}^2$; however some of these solutions coincide in $E \times E$. Thus the number of k-periodic points in $E \times E$ is equal to the cardinality of Λ^2/Λ_k, where

$$\Lambda_k = \{w \in \Lambda^2 | (M^k - I)^{-1} w \in \Lambda^2\} = (M^k - I)\Lambda^2.$$

Call $A_k = M^k - I$; if we fix a basis v_1, v_2 of Λ, the matrix of the action of A_k on Λ^2 with basis $(v_1, 0), (0, v_1), (v_2, 0), (0, v_2)$ is

$$B_k = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}.$$

Since $\det B_k = (\det A_k)^2 \cong c\lambda^{2k}$ (λ being the maximum eigenvalue of M), the number of k-periodic points grows as $|\Lambda^2/\Lambda_k| = \det B_k \cong c\lambda^{2k} = c\lambda_1(\phi_M)$.
If M is unipotent, we have seen in lecture 1 that ϕ_M preserves an elliptic fibration $\pi \colon E \times E \to E'$, and that the action on the base is the identity. It is not difficult to see that ϕ_M acts as a translation on the fibres of π, and that the action is periodic on a countable number of fibres. Therefore, ϕ_M^k admits an infinite number of periodic points.

Problem 6. a) *Let $f \colon X \to X$ be a zero entropy automorphism. Show that we can define polynomial analogues of the dynamical degrees*

$$d_p(f) = \limsup_{n \to +\infty} \frac{\log \|(f^n)^*_p\|}{\log n} \in \mathbb{N}.$$

The eigenvalues of the $f_p^* \colon H^{2p}(X, \mathbb{C}) \to H^{2p}(X, \mathbb{C})$ are algebraic integers; since the entropy is zero, they have to have modulus ≤ 1, hence, by Kronecker's lemma they are roots of unity and for some iterate of f they are all equal to 1 (i.e. f^* is unipotent). In this case we can explicitly show that the coefficients grow polynomially, hence the polynomial dynamical degree exists (and it is equal to the maximal dimension of a Jordan block minus 1).

b) *(∗) Does this work for birational transformations?*

Nobody knows!

c) *Let* $\dim(X) = 3$*; show that* $d_1(f) \leq 4$.

Idea: work by contradiction, and fix a base $v_1, \dots v_k$ ($k = d_1(f) + 1 \geq 6$) for a maximal Jordan block for f_1^*. We can cut X with a hyperplane section and apply Hodge's index theorem to show that, if $v, w \in H^{1,1}(X)$ are non-collinear, then $v.v, v.w, w.w$ cannot be simultaneously 0. By computing $(f^n)^* v_k . (f^n)^* v_k$ and $(f^n)^* v_{k-1} . (f^n)^* v_{k-1}$, we find that $d_2(f) \geq 2d_1(f) - 3$; since $d_2(f) = d_1(f^{-1})$, the same proof shows that $d_2(f^{-1}) = d_1(f) \geq 2d_1(f^{-1}) - 3 \geq 4d_1(f) - 9$, contradiction.

d) *(∗) Prove or give a counterexample: suppose that $\phi : X \dashrightarrow X$ is a dominant rational map. Then $\lambda_i(\phi)$ is an algebraic integer.*

This is an open question. In the case that ϕ is a morphism, it is certainly true: the $\lambda_i(\phi)$ are eigenvalues of a matrix with integral entries, and hence certainly integral. In general, however, little is known.

References

[Dem97] Jean-Pierre Demailly, *Complex analytic and differential geometry*, Citeseer, 1997.

[Les15] John Lesieutre, *Derived-equivalent rational threefolds*, Int. Math. Res. Not. IMRN (2015), no. 15, 6011–6020.

Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #3: Counterexamples in birational geometry
John Lesieutre and Federico Lo Bianco
July 21, 2016

Problem 1. *Let E be an elliptic curve; the Kummer surface X of E is the $K3$ surface obtained as the minimal resolution of the quotient $E \times E/-id$. Let $M \in SL_2(\mathbb{Z})$ be an infinite order semi-simple matrix.*

a) *Show that M induces an automorphism $\phi\colon X \to X$.*

We have the following commuting diagram

$$\begin{array}{ccc} Y & \xrightarrow{\rho_1} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ E \times E & \xrightarrow{\rho} & E \times E/-id \end{array}$$

where Y is the blow-up of $E \times E$ along the 16 two-torsion points in $E \times E$. Since the action of M on $E \times E$ preserves the two-torsion points, M induces an automorphism ψ_M on Y (the action on the exceptional divisors being given by the differential of ϕ_M); since $\psi_M(-p) = -\psi_M(p)$, this automorphism descends to an automorphism ϕ of X.

b) *Show that the strict transform C of the curve $E \times \{0\}$ is a (-2)-curve in X.*

Once we know that X is a $K3$ surface, we can just apply adjunction formula to get

$$K_C = (K_X + \mathcal{O}_X(C))|_C = \mathcal{O}_C(C)$$

and

$$C.C = \deg(\mathcal{O}_C(C)) = \deg K_C = -2$$

since $C \cong \mathbb{P}^1$.

One can also explicitly write ρ_1 in local coordinates around the exceptional divisors for π_1 to compute $C.C = -2$. In a similar fashion one can also show that X is actually a $K3$ surface: indeed, since ρ_1 is $2:1$ outside the exceptional divisors E_i for π_1 and $1:1$ on the E_i-s, when pulling back a meromorphic 4-form α on X to Y the E_i-s appear as simple zeroes together with the pull-back of zeroes of α; therefore,

$$K_Y = \sum E_i = \rho_1^* K_X + \sum E_i,$$

thus $K_X = 0$.

c) *Show that there exists an infinite number of (-2)-curves whose classes in $N^1(X)$ are distinct (hint: take C and all its iterates).*

Let us first show that all the images of C by ϕ are distinct: indeed, let E_0 be the exceptional divisor relative to the (image of the) point $(0,0)$. The curve C intersects E_0 only at the point p relative to the horizontal direction; since M is semi-simple of infinite order, the eigenspaces are not defined over $\mathbb{Q}$, so the orbit of p is infinite and contained into E_0. This shows that if $m \neq n$, then $\phi^m(C)$ and $\phi^n(C)$ intersect E_0 in two different points, thus in particular $\phi^m(C) \neq \phi^n(C)$. All the images of C are (-2)-curves since ϕ^* preserves the intersection product; since curves with negative self-intersection are fixed, their classes in $N^1(X)$ are distinct as well.

d) *Let E be an elliptic curve with an order 6 automorphism. Show that $(E \times E)/\tau$ is a rational surface.*

One way to do this is fairly direct, based on calculating the geometric invariants of X and applying Castelnuovo's criterion for detecting rational surfaces.

We take the indirect route. First, I claim that the surface X is uniruled. The quotient $(E \times E)/\tau$ has singularities of types $1/2(1,1)$, $1/3(1,1)$, and $1/6(1,1)$. Let X be its minimal resolution. Since $E \times E \to (E \times E)/\tau$ is étale in codimension 1, a computation of the discrepancies (i.e. the coefficients of the exceptional divisors in the resolution) gives $K_X \sim -\frac{1}{3}E_3 - \frac{2}{3}E_6$, where E_3 is the sum of the exceptional divisors over the singularities of type $1/3(1,1)$, and E_6 is the exceptional divisor over the unique singularity of type $1/6(1,1)$. In particular, $-K_X$ is effective, and this means that X is uniruled.

Notice as well that the surface X contains infinitely many (-1)-curves: arguing as in the first part of the problem, we find that the image of $E \times 0$ in the quotient is a (-1)-curve, and it has infinite order under the action of the automorphism group.

What uniruled surfaces are out there? According to the Enriques–Kodaira classification, they're all either rational, or blow-ups of $\mathbb{P}_C(\mathcal{E})$, where C is a curve of positive genus and $\mathcal{E}$ is a rank-2 vector bundle on E. The latter surfaces have only finitely many rational curves of negative self-intersection: any such curve must be contained in a fiber (since it can't dominate C), and only the finitely many reducible fibers can contain curves of negative self-intersection.

Problem 2. *Fill in the details in the derived categories example:*

a) *Let $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ be the blow-ups of $\mathbb{P}^n$ ($n \geq 3$) at two configurations of points. Show that $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ differ only by the action of $\mathrm{PGL}(n+1)$ and permutations of the points.*

I am going to cheat here and refer you to [Les15]; this is Lemma 1. Here's the idea: if you blow up $\mathbb{P}^2$ at a bunch of points, there are different ways to blow it back down. For example, you can contract the strict transform of a line between two of the points you blew up. In higher dimensions, this is not the case: if you blow up points in $\mathbb{P}^3$, there are no contractible divisors on the blow-up except the ones you started with.

b) *Prove that a very general configuration of 8 or more points in $\mathbb{P}^3$ has infinite orbit under Cremona transformations (up to the action of $PGL(4)$)*

This is Lemma 3 in the same paper. The gist is that you should cheat a little bit: choose your original configuration so that the points are slightly special, with four of them coplanar. Then you can show that the new configurations you get after Cremona transformations are also somehow special, but in different ways (e.g. there is a cubic that's double at four of the eight points, and passes through the other four – a codimension 1 condition on configurations).

Problem 3. *Suppose that $\phi : X \to X$ is a positive entropy automorphism of a surface, and D is the leading eigenvector of the action of ϕ^* on $N^1(X)$. Suppose that C is a ϕ-periodic curve. Show that $D \cdot C = 0$. Can you prove the converse?*

Since the automorphism preserves the intersection form, we have

$$D \cdot \phi^n(C) = (\phi^*)^n(D) \cdot C = \lambda^n D \cdot C.$$

The term on the left and $D \cdot C$ are both rational, while λ^n is irrational. So it must be that $D \cdot C = 0$.

In fact the converse is true as well. I'm going to use exercise 5(a) below, though this can probably be avoided through judicious application of the Hodge index theorem. Suppose that $D \cdot C = 0$, but that C is not ϕ-periodic. We have seen that $D \cdot \phi^n(C) = 0$ for all n, giving an infinite set of curves with $D \cdot C = 0$. According to 5(a), it must be that all but finitely many of these curves fit into a positive-dimensional family; in particular, the class of C is nef. But then $D^2 = 0$, $D \cdot C = 0$, and $C^2 \geq 0$, contradicting the Hodge index theorem since D and C are nef. (Is this true?)

Problem 4. *Let X be a smooth projective surface. If D is any pseudoeffective divisor, it admits aunique* Zariski decomposition *$D = P + N$, in which P is nef, N is effective, and $P \cdot N_i = 0$ for each component of N. One might try to generalize this to higher dimensional settings, but there is trouble...*

a) *Let X be the blow-up of $\mathbb{P}^2$ at 3 collinear points, and let $D = 3H - 2E_1 - 2E_2 - 2E_3$. What is the Zariski decomposition of D?*

Note that the divisor D is actually big: it's $H + 2(H - E_1 - E_2 - E_3)$. The second summand is effective, and the first is the pullback of a big divisor from $\mathbb{P}^2$, hence big itself.

Now, let $N = \frac{3}{2}(H - E_1 - E_2 - E_3)$ and $P = \frac{1}{2}(3H - E_1 - E_2 - E_3)$. Then $D = P + N$, and we have $P \cdot N = 0$ as required. Since N is just 3/2 the class of the strict transform of a line through the points, it's surely effective, and so I just need to convince you that P is nef. Write $2P = (H - E_1) + (H - E_2) + (H - E_3)$, and notice that each of these is basepoint free, so P is nef as claimed.

b) *Let D be the divisor with $\mathbf{B}_-(D)$ not Zariski-closed, discussed in lecture. Show that D can not be expressed in the form $D = P + N$ with P nef and N effective.*

The D there was constructed as the leading eigenvector of some matrix $M_\sigma : N^1(X) \to N^1(X)$. In particular, D must be an extremal ray on the pseudoeffective cone $\overline{\text{Eff}}(X)$. If $D = P+N$, then P and N must both be proportional to D (since both of these are required to be pseudoeffective). But D is not nef, so it can't be proportional to P.

c) *Show that there does not even exist a birational model $\pi : Y \to X$ and a decomposition $\pi^* D = P + N$, with P nef and N effective.*

Sometimes, in higher dimensions, one can find an analog of the Zariski decomposition after passing to some higher birational model. For example, let X be the blow-up of $\mathbb{P}^3$ at two points, and let D be the strict transform of the plane between those two points. This divisor is not nef, since it has interscetion -1 with the strict transform of the line that goes through the two points. D can not be written as $P + N$ with P nef and N effective, by the same argument as in the previous part.

However, let $\pi : Y \to X$ be the blow-up along the line. Then $\pi^* D = P + N$, where N is the exceptional divisor of π, and D is the strict transform of the plane. This strict transform is now basepoint free, hence nef.

Sadly, this is not always to be. Let X and D be as in the problem. Then $D \cdot C = 0$ for an infinite set of curves. Let $\pi : Y \to X$ be any birational map from a smooth variety Y, and suppose that $\pi^* D = P + N$ where N is effective. I claim that P can't be nef. Indeed, pushing forward, we have $D = f_* P + f_* N$. Since D is extremal, both of these classes must be proportional to D. Since N is effective and $\mathbb{R}_{\geq 0} D$ has no effective representative, it must be that $f_* N = 0$, so that the divisor N is π-exceptional. Now $C \subset X$ be a curve with $D \cdot C < 0$ and which is not contained in the image of the exceptional divisors of Y (which is a finite union of curves, so we can certainly find such a C). Let $\tilde{C}$ be the strict transform of C on Y. Then $N \cdot \tilde{C} > 0$ since $\tilde{C}$ is not contained in the exceptional divisor. But this gives $P \cdot C = D \cdot C - N \cdot C < 0$, so P can't be nef.

Problem 5. a) *Suppose that X is a surface and D is a nef class on X. Show that the number of curves with $D \cdot C = 0$ is either finite or uncountable.*

I'm going to punt on this one as well: you can find this as Lemma 3.1 in [Tot09].

b) *Let $X = \mathrm{Bl}_8 \mathbb{P}^3$. Show that $-K_X$ is nef, but there exists an infinite discrete set of curves on X with $-K_X \cdot C = 0$.*

This is the main result of [LO16]. The construction is based on using Cremona transformations on the blow-up; it's pretty similar to the derived category example we discussed in lecture.

Problem 6. *Find a big $\mathbb{R}$-divisor with non-closed $\mathbf{B}_-(D)$.*

We saw in lecture that there exists a smooth threefold X and a pseudoeffective divisor D on X such that there is a Zariski dense set of curves C_n with $D \cdot C_n < 0$.

Now, let $Y = \mathbb{P}_X(\mathcal{O} \oplus \mathcal{O}(1))$, where $\mathcal{O}(1)$ is a very ample divisor on X. There are two maps on Y: first, there is $\pi : Y \to X$, the $\mathbb{P}^1$-bundle structure. Second, there is $f : Y \to CX$, the map to the cone over X obtained by contracting the negative section of Y. Let $i : X \to Y$ be the inclusion of the negative section, and set $D' = f^* H + \pi^* D$, where H is ample on CX.

Since H is ample, $f^* H$ is big and nef. Since D is pseudoeffective, so too is $\pi^* D$. It follows that the divisor D' is big. Moreover, $f^* H \cdot i(C_n) = 0$ since $i(C_n)$ is contracted by f. Thus $D' \cdot i(C_n) = D \cdot C_n < 0$, and so D' is also negative on a countably set of curves.

This means that each $i(C_n)$ is contained in $\mathbf{B}_-(D')$, but we still need to explain why this set is not Zariski closed. Since the $i(C_n)$ are dense in a divisor, it's sufficient to prove that $\mathbf{B}_-(D')$ does not contain any divisor. This follows from the fact that $f^* H$ and $\pi^* D$ are both movable classes, together with the results of Nakayama [Nak04].

References

[Les15] John Lesieutre, *Derived-equivalent rational threefolds*, Int. Math. Res. Not. IMRN (2015), no. 15, 6011–6020.

[LO16] John Lesieutre and John Christian Ottem, *Curves disjoint from a nef divisor*, Michigan Math. J. **65** (2016), no. 2, 321–332.

[Nak04] Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[Tot09] Burt Totaro, *Moving codimension-one subvarieties over finite fields*, Amer. J. Math. **131** (2009), no. 6, 1815–1833.

Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #4: Invariant fibrations and classification problems
John Lesieutre and Federico Lo Bianco
July 22, 2016

Problem 1. *Let $\psi : S \to S$ be a positive entropy automorphism of a rational surface, and let $\phi : S \times C \to S \times C$ be given by $\psi \times \mathrm{id}$, where C is a curve. Compute $\lambda_1(\phi)$ and $\lambda_2(\phi)$.*

Let $\pi\colon S \times C \to C$ be the natural projection and let $\lambda_p(\phi|\pi)$ denote the relative dynamical degrees; since the action on the base C is the identity, we have $\lambda_p(\phi|\pi) = \lambda_p(\psi)$. By Dinh-Nguyen [DN11] we have

$$\lambda_1(\phi) = \max\{\lambda_1(id_C), \lambda_1(\phi|\pi)\} = \max\{1, \lambda_1(\psi)\} = \lambda_1(\psi)$$

and, using exercise 1 of lecture 2 and the Theorem on surfaces in lecture 1, we get

$$\lambda_2(\phi) = \lambda_1(\phi^{-1}) = \lambda_1(\psi^{-1}) = \lambda_1(\psi).$$

Problem 2. *Suppose that $\dim X = 3$, $\phi : X \dashrightarrow X$ is birational, and $\pi : X \dashrightarrow Y$ is a map to a surface with $\pi \circ \phi = \psi \circ \pi$. Prove that $\lambda_1(\phi) = \lambda_2(\phi)$.*

Since the fibres of π are 1-dimensional, $\lambda_0(\phi|\pi) = \lambda_1(\phi|\pi) = 1$. By Dinh-Sibony we have

$$\lambda_1(\phi) = \max\{\lambda_1(\psi), \lambda_1(\phi|\pi)\} = \lambda_1(\psi)$$

and

$$\lambda_2(\phi) = \lambda_1(\phi^{-1}) = \lambda_1(\psi^{-1}) = \lambda_1(\psi).$$

Problem 3. *Let $f\colon X \dashrightarrow X$ be an automorphism of a surface with positive entropy. Show that the orbit of a very general point is Zariski-dense.*

If this wasn't the case, then by Amerik-Campana [AC08], f would either be of finite order (contradiction), or preserve a fibration onto a curve, contradicting the Theorem on surfaces in lecture 1.

Problem 4. *Suppose that X is a variety of general type (i.e. K_X is big). Show that $\mathrm{Bir}(X)$ is finite (you may assume that we are working over a field of characteristic 0).*

This result is originally due to Matsumura. By definition, there's some m so that $|mK_X|$ gives a map $\phi_{|mK_X|} : X \to \mathbb{P}^N$ that is birational onto its image M. Then $\mathrm{Bir}(X) = \mathrm{Aut}(M)$, which is a subset of $\mathrm{PGL}(|mK_X|^*)$. The image is exactly the set of things the equations of the subvariety defining M, and so it's a closed algebraic subgroup of $\mathrm{PGL}(|mK_X|^*)$.

Either this subgroup is 0-dimensional, in which case we're done, or it's positive-dimensional, in which case we're going to get a contradiction with the fact that X is general type. If it's positive dimensional, it has a one-dimensional algebraic subgroup G, which must be either $\mathbb{G}_a$ or $\mathbb{G}_m$ (you can find a discussion of this in MO thread #143203).

Now, if x is any point on X, we can look at the orbit $G \cdot x$. This shows that there's a rational curve through every point on X, and in fact these curves belong to an algebraic family. By a standard Hilbert scheme argument, there's a dominant rational map $X' \to X$, where X' admits a map $X' \to Y$ with general fibers $\mathbb{P}^1$: here X' is a component of the universal family over the Hilbert scheme, and Y is the component of the Hilbert scheme parametrizing our rationals curves on X. The map $X' \to X$ is separable, since we're in characteristic 0, and this means that if we pullback a nonzero section of mK_X, we get a nonzero section of $mK_{X'}$. But by repeated application of adjunction, this would yield a holomorphic 1-form on $\mathbb{P}^1$, which is impossible.

Problem 5. *Suppose that $\pi : X \to \mathbb{P}^3$ is the blow-up at some set of points. Show that X does not admit any automorphism of positive entropy. [Hint: let E be one of the exceptional divisors, and suppose it has infinite orbit under $\phi : X \to X$. Can the $\phi^n(E)$ intersect?]*

We will actually show that X can't have any automorphisms of infinite order, except possibly the lifts of linear maps from $\mathbb{P}^3$, which have entropy 0. Suppose that $\phi : X \to X$ is an infinite order automorphism. Let E be one of the exceptional divisors of the automorphism. Either E is ϕ-periodic, or E has infinite orbit under ϕ.

If E is ϕ-periodic, then we can replace ϕ by some iterate and assume that E is fixed; since ϕ has infinite order, this iterate is not the identity map. Now, let $X \to Y$ be the blow-down of E. Since E is ϕ-invariant, the map ϕ descends to an automorphism of Y. In this case, we replace X with Y and start from the beginning.

As a result, we may suppose that E is not periodic under ϕ, so that the divisors $\phi^m(E)$ are all distinct.

Problem 6. *Let $X = \mathrm{Hilb}^n(S)$ be the Hilbert scheme of points of a $K3$ surface (i.e. the minimal resolution of the symmetric product $\mathrm{Sym}^n(S)$. Then X is a $2n$-dimensional irreducible symplectic holomorphic (or hyperkähler) manifold, and a result by Verbitsky says that the cup-product induces injections*

$$S^p(H^2(X, \mathbb{C})) \hookrightarrow H^{2p}(X, \mathbb{C})$$

for $p = 1, \ldots, n$.
Let $f : S \to S$ be a positive entropy automorphism with $\lambda_1(f) = \lambda > 1$. Show that f induces an automorphism of X and compute its dynamical degrees.

The product $f \times f \times \cdots \times f$ (n times) clearly induces an isomorphism of $Sym^n(S)$; the Hilbert scheme is obtained by resolving $Sym^n(S)$ along the singular set, which is the image of the set $\Delta = \{(x_1, \ldots, x_n) \in S \times \cdots \times S | x_i = x_j \text{ for some } i \neq j\}$. Since Δ is preserved by $f \times f \times \cdots \times f$, we have an induced automorphism g of $\mathrm{Hilb}^n(S)$.
Since the natural projection $S^n \dashrightarrow \mathrm{Hilb}^n(X)$ is generically finite, by exercise 1 of lecture 2 we have

$$\lambda_p(g) = \lambda_p(f \times \cdots \times f)$$

and in particular $h_{\rm top}(g) = h_{\rm top}(f \times \cdots \times f)$.
It is not hard to see that

$$h_{\rm top}(f \times \cdots \times f) = nh_{\rm top}(f).$$

Indeed, two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots y_n)$ define (N, ϵ)-separated orbits if there exists i such that x_i and y_i define (N, ϵ)-separated orbits, and conversely, if $\mathbf{x}$ and $\mathbf{y}$ define (N, ϵ)-separated orbits, then for some i, x_i and y_i define $(N, \epsilon/\sqrt{n})$- separated orbits. This allows to give a precise estimate of the number of (N, ϵ)-separated orbits for $f \times \cdots \times f$.
Now, by Verbitsky's result we have $\lambda_p(g) = \lambda_1(g)^p$ for $p = 1, \ldots, n$: indeed, if $v \in Pic(X) \subset H^2(X, \mathbb{Z})$ is an eigenvector for g^* with maximal eigenvalue $\lambda_1(g)$, then $v^p \in H^{2p}(X, \mathbb{Z})$ is a (non-trivial) eigenvector for g^* with eigenvalue $\lambda_1(g)^p$, so that $\lambda_p(g) \geq \lambda_1(g)^p$; the converse inequality follows from log-concavity. Furthermore, the same proof applied to g^{-1} shows that $\lambda_{2n-p}(g) = \lambda_1(g)^p$ for $p = 1, \ldots, n$.
In conclusion, we have

$$nh_{\rm top}(f) = h_{\rm top}(g) = \log(\lambda_n(g)) = n \log \lambda_1(g)$$

so that $\lambda_1(f) = \lambda_1(g)$ and

$$\lambda_p(g) = \begin{cases} \lambda_1(f)^p & p = 1, \ldots, n \\ \lambda_1(f)^{2n-p} & p = n+1, \ldots, 2n \end{cases}$$

Problem 7. *Let E be an elliptic curve with an order 6 automorphism τ.*

a) *Show that $(E \times E \times E)/\tau$ admits an imprimitive automorphism of positive entropy (hint: if the automorphism is induced by M, what will the dynamical degrees be? Find a matrix so that $\lambda_1(\phi) \neq \lambda_2(\phi)$.*

For this one, I'll refer you to [OT15], where you can find it as Lemma 4.3. The paper has a nice discussion of using relative dynamical degrees to show that there is no invariant fibration.

b) *For what values of n is E^n/τ uniruled? (*) Rational?*

There's a good way to check whether something like this is uniruled, using the BDPP theorem: we need to find a model $\pi : X \to E^n/\tau$ with terminal singularities, and then try to figure out whether the canonical class is pseudoeffective or not. In particular, if we can show that the $-K_X$ is effective, then the variety is uniruled. On the other hand, if K_X is effective, then it isn't.

We still have to find X and compute the canonical class, however. The singular points on the quotient are all isolated, and they come from the fixed points of the action of τ on E^n. Our variety has three different kinds of singularities, depending on the stabilizers of the point: they can have stabilizer of order 2, 3, or 6.

We need to make a standard calculation of how the canonical class changes when we resolve these singularities by blowing up (a good reference for this sort of thing is [Rei87]). Let ω be a dth root of unity acting on $\mathbb{C}[x, y, z]$ by multiplication by ω in each variable (we'll take $d = 2, 3, 6$, depending on which point we want). The quotient is (affine locally) Spec of the ring of invariants, which is generated by monomials $x^i y^j z^k$ with $i + j + k = d$. This is isomorphic to the cone over the dth Veronese embedding $\mathbb{P}^{n-1} \to \mathbb{P}^N$.

Let P be the cone in question. Let $\pi : C = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(d)) \to P$ be resolution obtained by blowing up the cone point, with exceptional divisor $E \cong \mathbb{P}^{n-1}$ that has normal bundle $\mathcal{O}(-(n-1))$. Then $K_C = \pi^* K_P + aE$ for some a. By adjunction, $K_E = (a+1)E|_E$, and we have seen that $K_E = \mathcal{O}_{\mathbb{P}^2}(-3)$ and $E|_E = \mathcal{O}_{\mathbb{P}^2}(-d)$. Putting this together, we obtain $a = \frac{n}{d} - 1$. When $d = 2$ this is positive as long as $n \geq 3$, so the singularity in question is terminal. We don't need to blow it up to obtain our X. When $d = 3$ this is non-negative for $n = 3$ and positive for $n \geq 4$. The worst singularities come from $d = 6$. Once $n \geq 6$, the discrepancy is non-negative, and K_X is effective, so the variety is of Calabi–Yau type. For $3 \leq n \leq 5$, after blowing up the non-terminal singularities, we obtain a terminal variety on which $-K_X$ is effective. This shows that X is uniruled.

The question of which of these varieties are rational has been an active topic lately. For $n = 3$ this is proved in [OT15]. For $n = 4$ it's known to be unirational [COV].

Problem 8. *Consider a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of general hypersurfaces of type $(1,1)$, $(1,1)$, $(2,2)$. Check that X is a Calabi-Yau threefold of Picard rank 2. Construct two involutive pseudoautomorphisms on X, and find a curve in the indeterminacy locus of each. Show that the composition has $\lambda_1 > 1$.*

This example is from [Ogu14], where you can find a detailed description of many of the nice aspects of the geometry of this variety. This example appears as Example 6.1.

Problem 9. *Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold, and that $E \subset X$ is a smooth divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that for any nonzero n, $E \cap \phi^n(E)$ is a union of rulings of E.*

Let D be a dominant eigenvector of the pullback map $\phi^* : N^1(X) \to N^1(X)$ (i.e. an eigenvector with the largest eigenvalue). We saw during the lectures that this eigenvalue is a real number, which is greater than 1 by hypothesis. In fact, it must also be irrational: the matrix for ϕ^* has integer entries, as does its inverse. As a result, the only possible rational roots of the characteristic polynomial are -1 and 1.

Now, we compute a certain three-way intersection in two different ways.

$$\begin{aligned} D \cdot E \cdot \phi^n(E) &= (D \cdot \phi^n(E))|_E \\ &= \phi^{n*} D \cdot \phi^{n*} E \cdot \phi^{n*}(\phi^n(E)) \\ &= \lambda^n D \cdot \phi^{-n}(E) \cdot E \\ &= \lambda^n (D \cdot \phi^{-n}(E))|_E \end{aligned}$$

The quantity on the right of the first row is a rational number, since it's the intersection of two Cartier divisors on a smooth surface. The quantity on the right of the bottom is λ^n times a rational number. The only possibility is that $D \cdot E \cdot \phi(E) = 0$.

Now, D is nef, and so $D \cdot E$ is also nef. A Hodge index theorem argument shows that $D \cdot E$ is nonzero. Hence $D|E$ must be (numerically) a multiple of one of the rulings on E. The fact that $D \cdot E \cdot \phi^n(E) = 0$ then implies that $\phi^n(E)$ must be (again numerically) a multiple of the same ruling, for every value of n. But the only curve classes numerically equivalent to a ruling are rulings, and so $E \cdot \phi^n(E)$ must be a union of rulings on E for every value of n.

References

[AC08] Ekaterina Amerik and Frédéric Campana, *Fibrations méromorphes sur certaines variétés à fibré canonique trivial*, Pure Appl. Math. Q. **4** (2008), no. 2, part 1, 509–545. MR 2400885

[COV] Fabrizio Catanese, Keiji Oguiso, and Alessandro Verra, *On the unirationality of higher dimensional ueno-type manifolds*, arxiv:1506.01925.

[DN11] Tien-Cuong Dinh and Viêt-Anh Nguyên, *Comparison of dynamical degrees for semi-conjugate meromorphic maps*, Comment. Math. Helv. **86** (2011), no. 4, 817–840. MR 2851870

[Ogu14] Keiji Oguiso, *Automorphism groups of Calabi-Yau manifolds of Picard number 2*, J. Algebraic Geom. **23** (2014), no. 4, 775–795.

[OT15] Keiji Oguiso and Tuyen Trung Truong, *Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 1, 361–385.

[Rei87] Miles Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.